

# Crack avalanches in the three-dimensional random fuse model

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## Abstract

We analyze the scaling of avalanche precursors in the three-dimensional random fuse model by numerical simulations. We find that both the integrated and non-integrated avalanche size distributions are in good agreement with the results of the global load sharing fiber bundle model, which represents the mean-field limit of the model.

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## 1. Introduction

Understanding the scaling properties of fracture in disordered media represents an intriguing theoretical problem with some technological implications [1]. Of particular interest is the acoustic emission (AE) recorded in a stressed material before failure. The noise is a consequence of micro-cracks forming and propagating in the material and provides an indirect measure of the damage accumulated in the system. For this reason, AE is often used as a non-destructive tool in material testing and evaluation.

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The distribution of crackle amplitudes follows a power law, suggesting an interpretation in terms of scaling theories. This behavior has been observed in several materials such as wood [2], cellular glass [3], concrete [4] and paper [5].

The statistical properties of fracture in disordered media are captured qualitatively by lattice models, describing the medium as a discrete set of elastic bonds with randomly distributed failure thresholds [1,6,7]. In the simplest approximation of a scalar displacement, one recovers the random fuse model (RFM) where a lattice of fuses with random thresholds are subject to an increasing external current [6,7]. Fracture of the RFM is preceded by avalanches of failure events [8–10] which are reminiscent of the acoustic emission activity observed in experiments. The distribution of avalanche sizes (i.e., the number of bonds participating in an avalanche) follows a power law. Initially, two-dimensional simulations yielded an exponent close to  $\tau = \frac{5}{2}$  [9], the value expected in the fiber bundle model (FBM) [11,12]. In that model load is redistributed equally to all the fibers, representing thus a sort of mean-field limit of the RFM [9]. More recently large-scale simulations, however, displayed significant (non-universal) deviations from the mean-field result [13]. Only some preliminary results are reported in the literature for three dimensions [14]. Here, we show that avalanches in the three-dimensional RFM follow quite closely the mean-field predictions. This is partly expected since normally scaling exponents tend to the mean-field limit as the lattice dimensionality increases, although the exact value for the upper critical dimension is not known for this problem.

## 2. The random fuse model

In the random thresholds fuse model [6,7], the lattice is initially fully intact with bonds having the same conductance, but the bond breaking thresholds,  $t$ , are randomly distributed based on a thresholds probability distribution,  $p(t)$ . The burning of a fuse occurs irreversibly, whenever the electrical current in the fuse exceeds the breaking threshold current value,  $t$ , of the fuse. Periodic boundary conditions are imposed in both of the horizontal directions to simulate an infinite system and a constant voltage difference,  $V$ , is applied between the top and the bottom of the lattice system bus bars.

Numerically, a unit voltage difference,  $V = 1$ , is set between the bus bars and the Kirchhoff equations are solved to determine the current flowing in each of the fuses. Subsequently, for each fuse  $j$ , the ratio between the current  $i_j$  and the breaking threshold  $t_j$  is evaluated, and the bond  $j_c$  having the largest value,  $\max_j i_j/t_j$ , is irreversibly removed (burnt). The current is redistributed instantaneously after a fuse is burnt implying that the current relaxation in the lattice system is much faster than the breaking of a fuse. Each time a fuse is burnt, it is necessary to re-calculate the current redistribution in the lattice to determine the subsequent breaking of a bond. The process of breaking of a bond, one at a time, is repeated until the lattice system falls apart. In this work, we assume that the bond breaking thresholds are distributed based on a uniform probability distribution, which is constant between 0 and 1.

Numerical simulation of fracture using large fuse networks is often hampered due to the high computational cost associated with solving a new large set of linear equations every time a new lattice bond is broken. Although the sparse direct solvers presented in Ref. [15] are superior to iterative solvers in two-dimensional lattice systems, for three-dimensional lattice systems, the memory demands brought about by the amount of fill-in during the sparse Cholesky factorization favor iterative solvers. Hence, iterative solvers are in common use for large-scale three-dimensional lattice simulations. The authors have developed an algorithm based on a block-circulant preconditioned conjugate gradient (CG) iterative scheme [16] for simulating three-dimensional random fuse networks. The block-circulant preconditioner was shown to be superior compared with the *optimal* point-circulant preconditioner for simulating three-dimensional random fuse networks [16]. Since the block-circulant and *optimal* point-circulant preconditioners achieve favorable clustering of eigenvalues (in general, the more clustered the eigenvalues are, the faster the convergence rate is), in comparison with the Fourier accelerated iterative schemes used for modeling lattice breakdown [17], this algorithm significantly reduced the computational time required for solving large lattice systems.

Using the algorithm presented in Ref. [16], we have performed numerical simulations on three-dimensional cube lattice networks. For many lattice system sizes, the number of sample configurations,  $N_{config}$ , used are extremely large to reduce the statistical error in the numerical results. In particular, we used  $N_{config} = 40\,000, 3840, 512, 128, 32$  for  $L = 10, 16, 24, 32, 48$ , respectively.

### 3. Avalanches

When the current is increased at an infinitesimal rate failure events cluster in the form of avalanches. The typical avalanche size increases with the current up to the last catastrophic failure event. The avalanche size distribution is a power law followed by an exponential cutoff at large sizes. The cutoff size  $s_0$  is increasing with the lattice size, so that we can describe the distribution by a scaling form

$$P(s, L) = s^{-\tau} g(s/L^D), \quad (1)$$

where  $D$  represents the fractal dimension of the avalanches. To confirm this finite size scaling assumption, we perform a data collapse imposing the mean-field exponent  $\tau = \frac{5}{2}$  and choosing  $D = 1.5$  (see Fig. 1). A direct power law fit of the distribution yields instead  $\tau = 2.55$ .

We have considered avalanche statistics integrating the distribution over all the values of the current, but the avalanche signal is not stationary: as the current increases so does the avalanche size. In Fig. 2 we report the distribution of avalanche sizes sampled at different values of the current  $I$ . For each sample, we normalize the current by its maximum value  $I_c$  and divide the  $I^* = I/I_c$  axis in 20 bins. We then

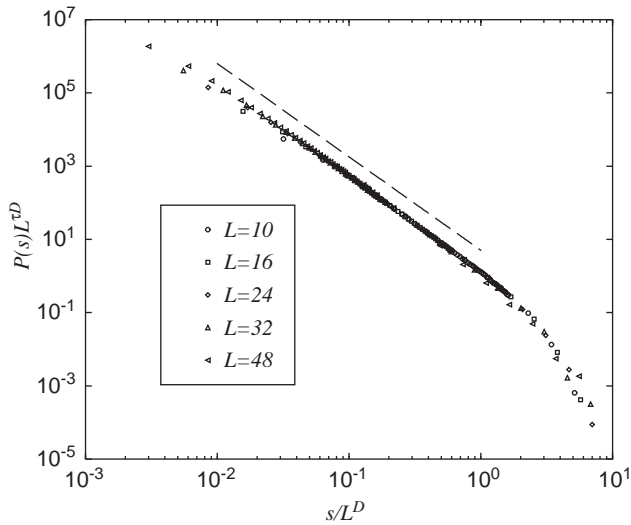


Fig. 1. Data collapse of the integrated avalanche size distributions. The exponents used for the collapse are  $\tau = 2.5$  and  $D = 1.5$ . The line with a slope  $\tau = 2.55$  is the best fit for the power law decay.

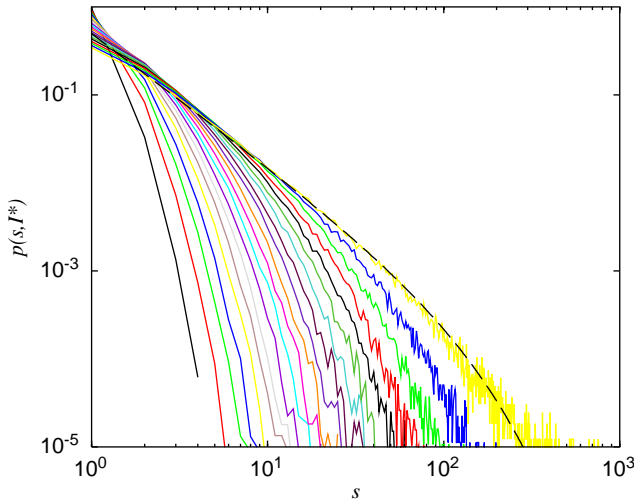


Fig. 2. The avalanche size distributions sampled over a small bin of the reduced current  $I^*$  for a cube lattice of size  $L = 48$ . The dashed line is a fit according to Eq. (2) with  $\gamma = \frac{3}{2}$ .

compute the avalanche size distribution  $p(s, I^*)$  for each bin and average over different realization of the disorder. The distribution follows a law of the type

$$p(s, I^*) = s^{-\gamma} \exp(-s/s^*), \tag{2}$$

with  $\gamma \simeq 1.5$  and  $s^*$  is an increasing function of  $I^*$ , in good agreement with mean-field results.

#### 4. Conclusions

We have performed numerical simulations of the RFM in three dimensions, focusing on the avalanche distributions. The scaling of the distributions is well captured by mean-field theory. This is in contrast with the behavior in two dimensions that shows larger deviations [13]. This can be expected on general grounds since typically the mean-field limit is approached as the dimensions are increased.

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