q-GAUSSIANS GALORE

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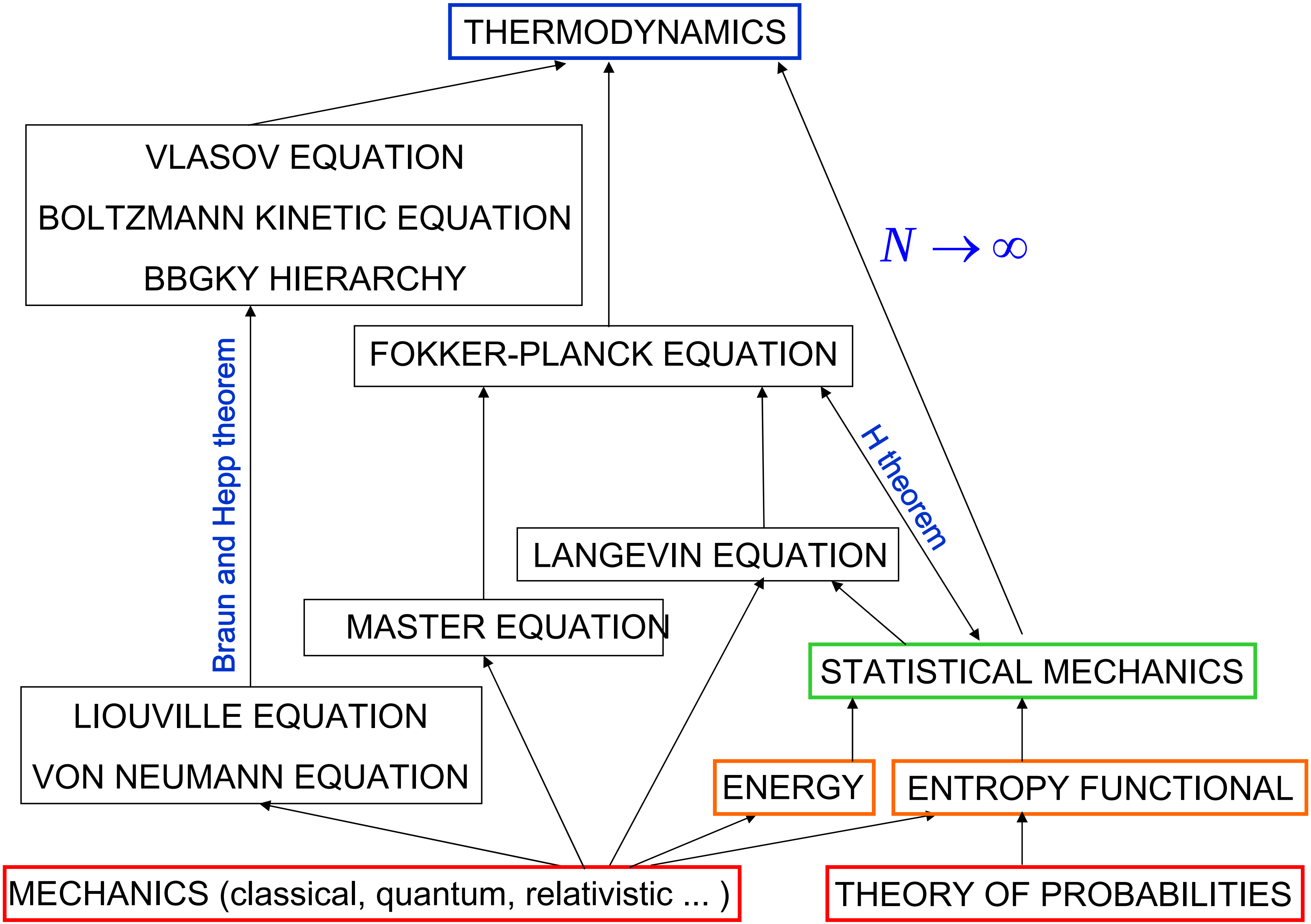
Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.


THERMODYNAMICS
VLASOV EQUATION
BOLTZMANN KINETIC EQUATION
BBGKY HIERARCHY
$N \to \infty$
FOKKER-PLANCK EQUATION
Braun and Hepp theorem
H theorem
LANGEVIN EQUATION
MASTER EQUATION
STATISTICAL MECHANICS
LIOUVILLE EQUATION
VON NEUMANN EQUATION
ENERGY
ENTROPY FUNCTIONAL
MECHANICS (classical, quantum, relativistic ...)
THEORY OF PROBABILITIES

POSTULATE FOR THE ENTROPIC FUNCTIONAL

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG entropy ($q = 1$)	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* (q *real*)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

Possible generalization of Boltzmann-Gibbs statistical mechanics

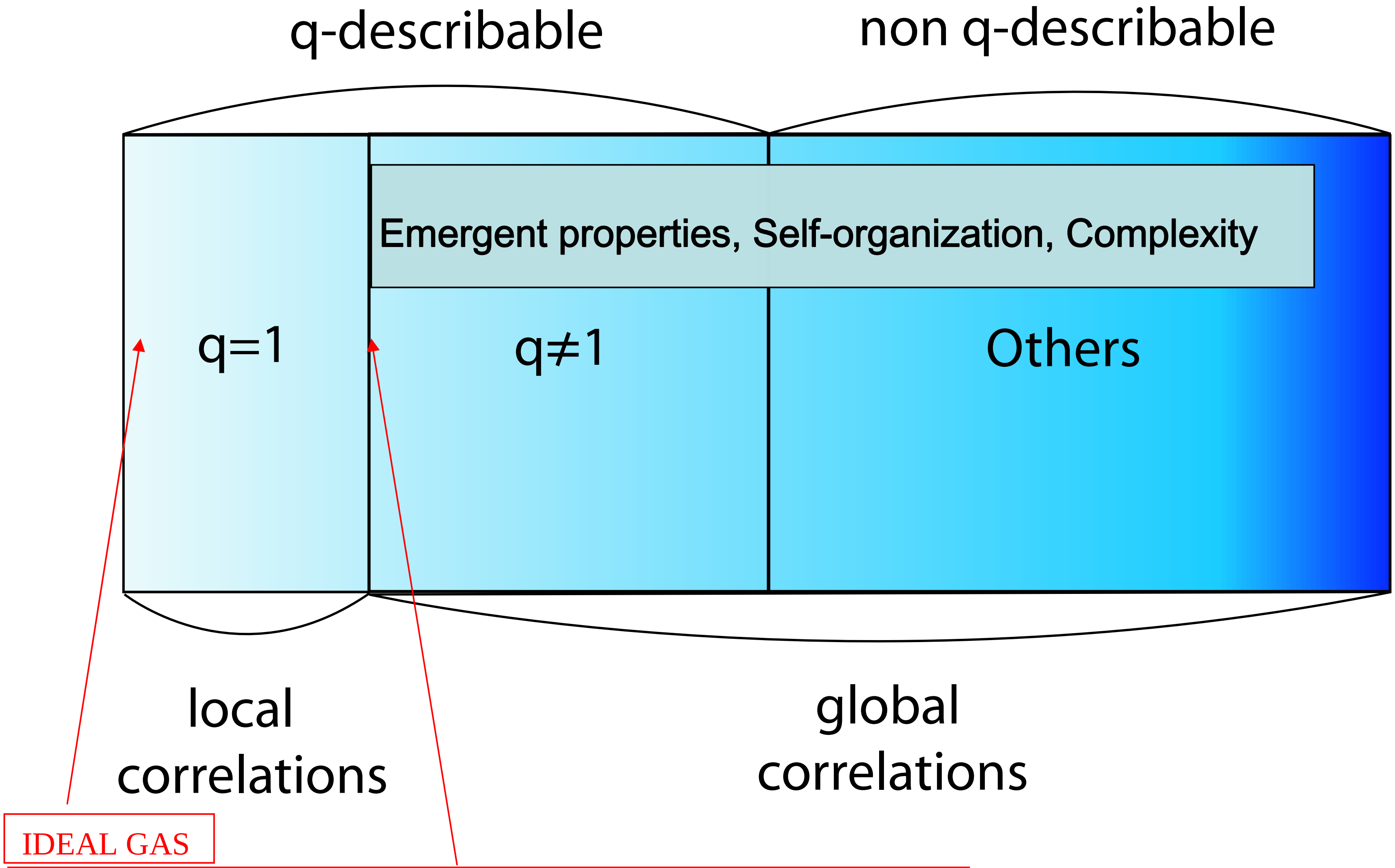
[C.T., J. Stat. Phys. **52**, 479 (1988)]

$DEFINITION\ (q-logarithm):$ $$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0)$$

$$\ln_1 x = \ln x$$

Hence, the entropies can be rewritten:

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k\ \ln W$	$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$
entropy S_q $(q \in R)$	$k\ \ln_q W$	$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$



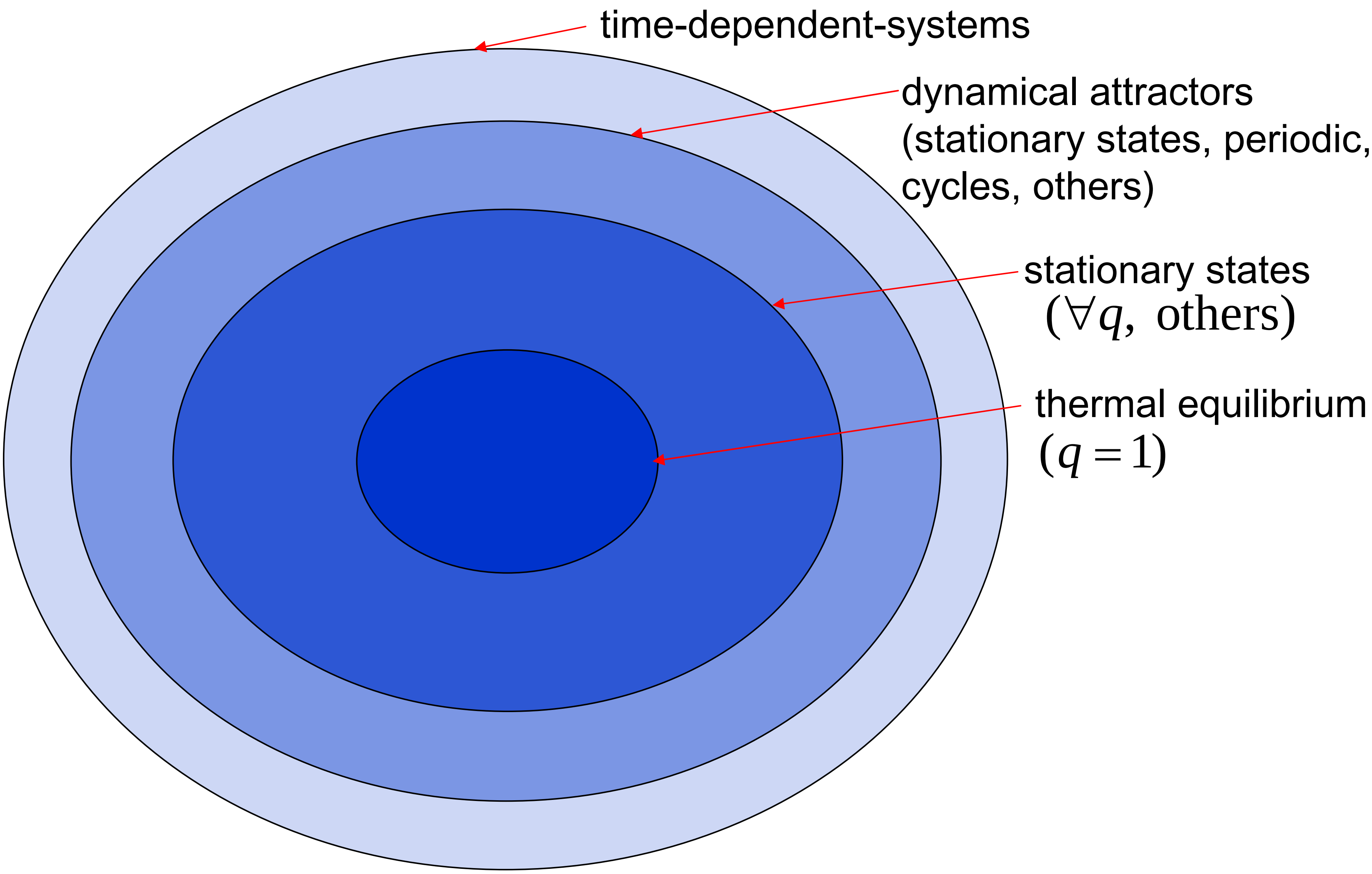
CRITICAL PHENOMENA

$$q = \frac{1+\delta}{2}$$ [A. Robledo, Mol Phys 103 (2005) 3025]

$$q = \frac{\sqrt{9+c^2}-3}{c}$$ [F. Caruso and C. T., Phys Rev E 78 (2008) 021101]

C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186
(European Physical Soc., 2005)

THERMOSTATISTICAL DYNAMICAL SYSTEMS:



GROUNDING STAT. MECH.: Entropy extremization

Extremization of S_q with appropriate constraints yields

$$p_q(x) \propto [1-(1-q)\beta\, x]^{\frac{1}{1-q}} \equiv e_q^{-\beta\, x}$$ (q-generalized Boltzmann weight)

[inverse function of $\ln_q x$]

If $\langle x \rangle = 0$, extremization of S_q yields instead

$$p_q(x) \propto [1-(1-q)\beta\, x^2]^{\frac{1}{1-q}} \equiv e_q^{-\beta\, x^2}$$ (q-generalized Gaussian)

STRONG CORRELATIONS:

HOW CAN THE NONADDITIVE ENTROPIC FUNCTIONAL S_q

BECOME AN EXTENSIVE ENTROPY ?

ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for any two probabilistically independent systems A and B,

$$S(A+B)=S(A)+S(B)$$

Therefore, since

$$S_q(A+B)=S_q(A)+S_q(B)+(1-q)\;S_q(A)\;S_q(B)\;,$$

S_{BG} and $S_q^{Renyi}\,(\forall q)$ are additive, and $S_q\;(\forall q\neq 1)$ is nonadditive .

EXTENSIVITY:

Consider a system $\Sigma\equiv A_1+A_2+...+A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N.

An entropy is extensive if

$$0<\lim_{N\to\infty}\frac{S(N)}{N}<\infty\;,\;\; i.e.,\;\; S(N)\propto N\;\;\;(N\to\infty)$$

HYBRID PASCAL - LEIBNITZ TRIANGLE

(N=0) $1\times\frac{1}{1}$

(N=1) $1\times\frac{1}{2}$ $1\times\frac{1}{2}$

(N=2) $1\times\frac{1}{3}$ $2\times\frac{1}{6}$ $1\times\frac{1}{3}$

(N=3) $1\times\frac{1}{4}$ $3\times\frac{1}{12}$ $3\times\frac{1}{12}$ $1\times\frac{1}{4}$

(N=4) $1\times\frac{1}{5}$ $4\times\frac{1}{20}$ $6\times\frac{1}{30}$ $4\times\frac{1}{20}$ $1\times\frac{1}{5}$

(N=5) $1\times\frac{1}{6}$ $5\times\frac{1}{30}$ $10\times\frac{1}{60}$ $10\times\frac{1}{60}$ $5\times\frac{1}{30}$ $1\times\frac{1}{6}$

$$\sum_{n=0}^{N}\binom{N}{n} r_{N,n} = 1 \quad (\forall N)$$

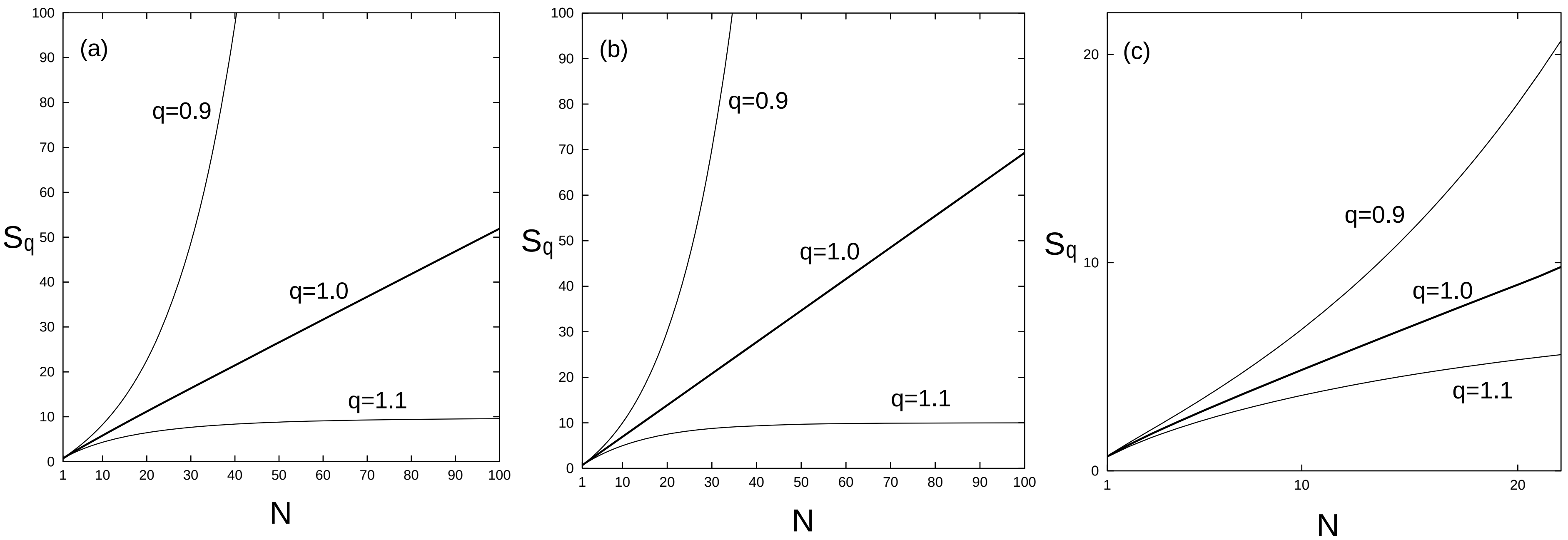
Blaise **Pascal** (1623-1662)
Gottfried Wilhelm **Leibnitz** (1646-1716)
Daniel **Bernoulli** (1700-1782)

$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$

I don't believe that atoms exist!

Ernst Mach (January 1897, Vienna)



Leibnitz triangle

$$\left(p_{N,0} = \frac{1}{N+1} \right)$$

N independent coins

$$\left(\begin{array}{l} p_{N,0} = p^N \\ \\ with\ p = 1/2 \end{array} \right)$$

Stretched exponential

$$\left(\begin{array}{l} p_{N.0} = p^{N^\alpha} \\ \\ with\ p = \alpha = 1/2 \end{array} \right)$$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$(N=0) \quad 1$

$(N=1) \quad 1/2 \quad 1/2$

$(N=2) \quad 1/3 \quad 1/6 \quad 1/3$

$(N=3) \quad 3/8 \quad 5/48 \quad 5/48 \quad 0$

$(N=4) \quad 2/5 \quad 3/40 \quad 1/20 \quad 0 \quad 0$

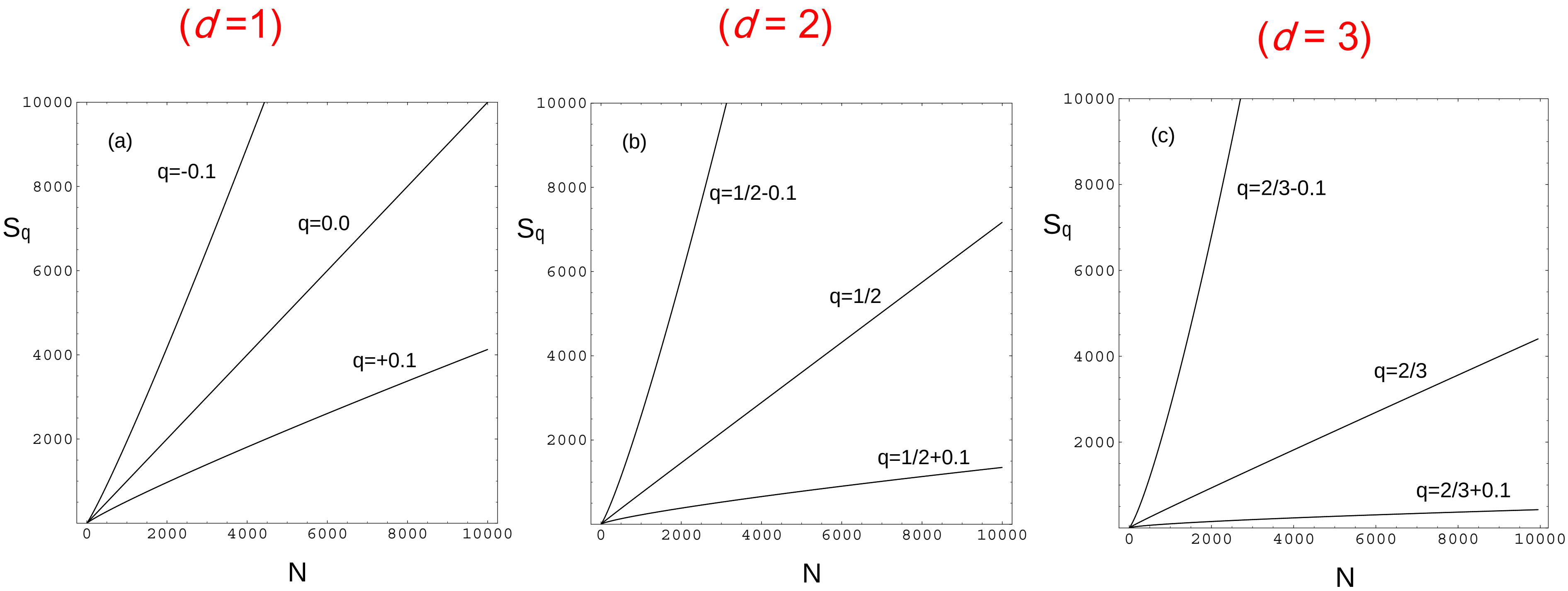
d+1

(It asymptotically satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

i.e., such that $S_q(N) \propto N \quad (N \to \infty)$



$$q = 1 - \frac{1}{d}$$

(All three examples asymptotically satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

PHYSICAL REVIEW E **78**, 021102 (2008)

Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

$$\hat{\mathcal{H}} = -\sum_{j=1}^{N-1} \left[(1+\gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1-\gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z\right]$$

$|\gamma| = 1 \quad \rightarrow \quad$ *Ising ferromagnet*

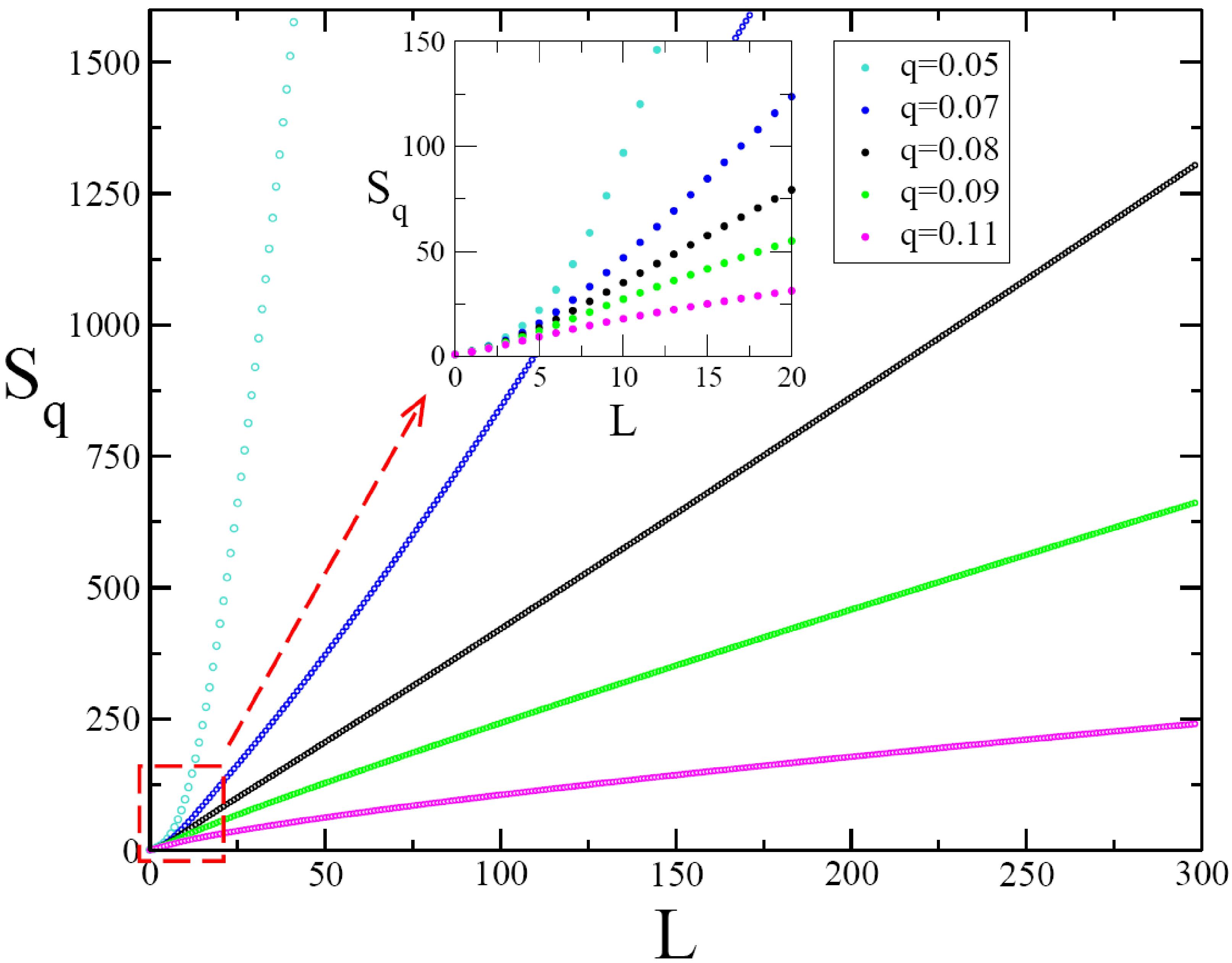
$0 < |\gamma| < 1 \quad \rightarrow \quad$ *anisotropic XY ferromagnet*

$\gamma = 0 \quad \rightarrow \quad$ *isotropic XY ferromagnet*

$\lambda \equiv$ *transverse magnetic field*

$L \equiv$ *length of a block within a* $N \rightarrow \infty$ *chain*

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)



F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

Using a Quantum Field Theory result

in P. Calabrese and J. Cardy, JSTAT P06002 (2004)

we obtain, at the critical transverse magnetic field,

$$q_{ent} = \frac{\sqrt{9+c^2}-3}{c}$$

with $c \equiv$ central charge in conformal field theory

Hence

Ising and anisotropic XY ferromagnets $\Rightarrow\ c = \frac{1}{2}\ \Rightarrow\ q_{ent} = \sqrt{37}-6\ \approx 0.0828$

and

Isotropic XY ferromagnet $\Rightarrow\ c = 1\ \Rightarrow\ q_{ent} = \sqrt{10}-3\ \approx 0.1623$

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

Summarizing, for a wide class of quantum problems, we know that

$$S_{BG}(N) \propto \ln L \propto \ln N \neq N \quad \text{for } d=1 \text{ quantum chains}$$

$$\propto L \propto \sqrt{N} \neq N \quad \text{for } d=2 \text{ bosonic systems}$$

$$\propto L^2 \propto N^{2/3} \neq N \quad \text{for } d=3 \text{ black hole}$$

$$\propto L^{d-1} \propto N^{(d-1)/d} \neq N \quad \text{for } d\text{-dimensional bosonic systems}$$

$$(d>1; \text{ area law})$$

$$\propto \frac{L^{d-1}-1}{d-1} \equiv \ln_{2-d} L \neq L^d \propto N \quad (d \geq 1) \quad \text{(NONEXTENSIVE!)}$$

For the same class of quantum problems, we expect

$$S_{q_{ent}}(N) \propto L^d \propto N \quad (d \geq 1;\ q_{ent} \neq 1) \quad \text{(EXTENSIVE!)}$$

(which we have illustrated for $d = 1, 2$)

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

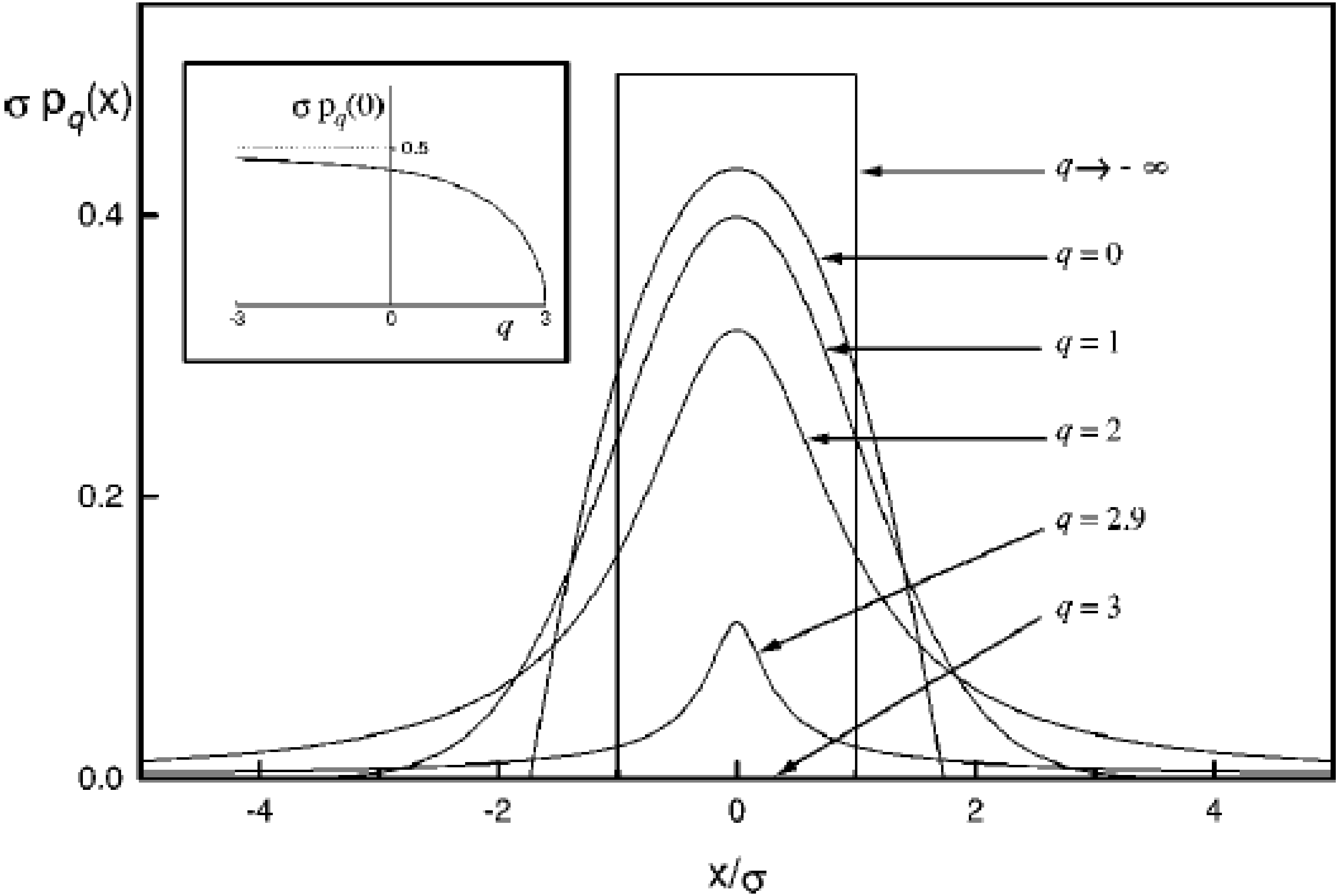
SYSTEMS	**ENTROPY S_{BG}** **(additive)**	**ENTROPY S_q (q<1)** **(nonadditive)**
Short-range interactions, weakly entangled blocks, etc	**EXTENSIVE**	**NONEXTENSIVE**
Long-range interactions (QSS), strongly entangled blocks, etc	**NONEXTENSIVE**	**EXTENSIVE**

STRONG CORRELATIONS:

HOW CAN THEY SUBSTANTIALLY MODIFY THE

CENTRAL LIMIT THEOREM AND ITS ATTRACTORS ?

q-GAUSSIANS: $p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \dfrac{1}{\left[1+(q-1)\,(x/\sigma)^2\right]^{\frac{1}{q-1}}}$ $(q<3)$



D. Prato and C. T., Phys Rev E **60**, 2398 (1999)

q – PRODUCT:

L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003)
E.P. Borges, Physica A **340**, 95 (2004)

The q - product is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

$Properties:$

$i)\quad x \otimes_1 y = x\,y$

$ii)\quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$ (extensivity of Sq)

$[whereas\ \ \ln_q (x\,y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$

(nonadditivity of Sq)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{ix\xi[f(x)]^{q-1}} f(x)\, dx$$

$(q \geq 1)$

(nonlinear!)

For q<1 see K.P. Nelson and S. Umarov, 0811.3777 [cs.IT]

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-independence:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$] *having zero* $q-$*mean values are said* q-*independent if*

$$F_q[X+Y](\xi) = F_q[X](\xi) \otimes_{\frac{1+q}{3-q}} F_q[Y](\xi) \quad ,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\ e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\ e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_{(1+q)/(3-q)} \left[\int_{-\infty}^{\infty} dy\ e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ h(x,y)\ \delta(x+y-z) = \int_{-\infty}^{\infty} dx\ h(x,z-x) = \int_{-\infty}^{\infty} dy\ h(z-y,y)$$

where $h(x,y)$ *is the joint density.*

$$q\text{-}\textit{independence means} \begin{cases} \textit{independence} & \textit{if } q=1\,,\ \textit{i.e., } h(x,y)=f_X(x)\,f_Y(y) \\ \textit{global correlation} & \textit{if } q\neq 1\,,\ \textit{i.e., } h(x,y)\neq f_X(x)\,f_Y(y) \end{cases}$$

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$-scaled attractor $\mathbb{F}(x)$ when summing $N \to \infty$ q-independent identical random variables with symmetric distribution $f(x)$ with $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \frac{1+q}{3-q} \right)$

	$q=1$ *[independent]*	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) *[globally correlated]*
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) =$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathbb{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x \rvert \ll x_c(q,2) \\ f(x) \sim C_q / \lvert x \rvert^{2/(q-1)} & \text{if } \lvert x \rvert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) =$ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \lvert x \rvert \ll x_c(1,\alpha) \\ f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{q,\alpha}$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\,\alpha}(x) \sim C^{*}_{q,\alpha} / \lvert x \rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad \textit{(intermediate regime)} \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\,2}(x) \sim C^{L}_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \quad \textit{(distant regime)} \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2009)

ON THE PHYSICAL MEANING OF *q* - INDEPENDENCE
(STRICT OR ASYMPTOTIC) SCALE INVARIANCE: IS IT NECESSARY? IS IT SUFFICIENT?

CANDIDATE MODELS FOR *q* - INDEPENDENCE:

1) *N* compact-support continuous variables with correlation introduced through a *N*-variate covariance matrix (strictly scale-invariant)
W. Thistleton, J.A. Marsh, K. Nelson and C. T., Cent Eur J Phys **7**, 387 (2009)
[see H.J. Hilhorst and G. Schehr, J Stat Mech (2007) P06003]

2) *N* binary variables with correlation introduced through the *q* - product
(strictly scale-invariant)
L.G. Moyano, C. T. and M. Gell-Mann, Europhys Lett **73** (2006) 813
[see H.J. Hilhorst and G. Schehr, J Stat Mech (2007) P06003]

3) *N* binary variables with correlation introduced through a family of triangles generalizing the Leibnitz one (strictly scale-invariant)
A. Rodriguez, V. Schwammle and C. T., J Stat Mech (2008) P09006
R. Hanel, S. Thurner and C. T., Eur. Phys. J. B (2009), in press

4) N-binary-discretized *q*-Gaussians (asymptotically scale-invariant)
A. Rodriguez, V. Schwammle and C. T., J Stat Mech (2008) P09006

STRONG CORRELATIONS:

HOW CAN THEY GENERATE MESOSCOPIC MEMORY ?

Strongly non-Markovian noise → Nonlinear homogeneous Fokker-Planck equation: A mesoscopic mechanism leading to nonextensive statistical mechanics

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial U(x)}{\partial x}\, p(x,t)\right] + D\,\frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} \quad [q<3;\ D(2-q)>0;\ t\geq 0]$$

Plastino and Plastino, Physica A **222** (1995) 347; Fuentes and Caceres, PLA **372** (2008)

1) The imposition of the H-theorem in this equation mandates the entropy to be S_q

V. Schwammle, E.M.F. Curado and F.D. Nobre,
Eur Phys J B **58** (2007) 159; Phys Rev E **76** (2007) 041123

2) Stationary state in the presence of any confining potential $U(x)$:

$$p(x,\infty) \propto e_q^{-\beta\,[U(x)-U(0)]}\text{, where } \beta > 0$$

3) If $U(x) = -k_1 x + \frac{1}{2} k_2 x^2 \quad (k_2 > 0)$, then

$$p(x,t) \propto e_q^{-\beta(t)\,x^2}\text{, where } 0 < \beta(\infty) < \infty$$

C T and DJ Bukman, Phys Rev E **54** (1996) R2197

4) If $U(x) = 0$, then $p(x,t) \propto e_q^{-\beta(t)\,x^2}$, where $\beta(t) \propto 1/t^{\frac{2}{3-q}}$,

hence x^2 scales like t^γ with $\gamma = \frac{2}{3-q}$ (prediction)

Multiplicative noise → Linear inhomogeneous Fokker-Planck equation: Another mesoscopic mechanism leading to nonextensive statistical mechanics

C. Anteneodo and C. T., J Math Phys **44** (2003) 5194

$$\frac{dx}{dt} = f(x) + g(x)\ \xi(t) + \eta(t)$$

[see also L. Borland, Phys Lett A **245** (1998) 67]

where $\xi(t)$ and $\eta(t)$ are independent zero-mean white Gaussian noises with amplitudes M and A. It follows (with Stratonovich choice)

$$\frac{\partial\, p(x,t)}{\partial\, t} = -\frac{\partial\, [\, f(x)\, p(x,t)]}{\partial\, x} + M\, \frac{\partial}{\partial x}\left\{ g(x) \frac{\partial [g(x) p(x,t)]}{\partial x} \right\} + A\, \frac{\partial^2 [p(x,t)]}{\partial\, x^2}$$

If the deterministic drift is proportional to the multiplicative-noise induced drift,

i.e., if $\quad f(x) = -\tau\, g(x)\, g'(x)$, $\qquad [e.g., f(x) \propto g(x) \propto x]$

then the stationary state is given by

↑ natural first physical choice

$$p(x,\infty) \propto e_q^{-\beta\, [g(x)]^2}$$

with $\quad q = \dfrac{\tau + 3M}{\tau + M} \geq 1 \qquad$ and $\quad \beta \equiv \dfrac{1}{kT} = \dfrac{\tau + M}{2A}$

Generalizable: $\quad q = \dfrac{\tau + 2M(2-\theta)}{\tau + 2M(1-\theta)} \geq 1 \quad$ and $\quad \beta \equiv \dfrac{1}{kT} = \dfrac{\tau + 2M(1-\theta)}{2A} \quad (0 \leq \theta \leq 1)$

$\theta = 0 \rightarrow$ Ito-Langevin equation; $\quad \theta = 1/2 \rightarrow$ Stratonovich-Langevin equation

B.C. Santos and C. T. (2009)

STRONG CORRELATIONS:

HOW CAN THEY BE OVERCOME BY FINITE-SIZE EFFECTS ?

ORDINARY DIFFERENTIAL EQUATION:

$$\frac{dy}{dx} = -ay^r - by^q \qquad (b \geq a \geq 0;\ q > r;\ y(0) = 1)$$

If $a = 0$ then $y = e_q^{-bx}$

If $b = 0$ then $y = e_r^{-ax}$

If $b > a > 0$ then

$$x = -\int_1^y \frac{du}{au^r + bu^q} = \frac{r}{b(-1+q)(q-r)} - \frac{\left(r + (q-r)\,{}_2F_1\left[\frac{-1+q}{q-r}, 1, 1 + \frac{-1+q}{q-r}, -\frac{a}{b}\right]\right)}{b(-1+q)(q-r)}$$

$$- \frac{ry^{1-q}}{b(-1+q)(q-r)} + \frac{y^{1-q}\left(r + (q-r)\,{}_2F_1\left[\frac{-1+q}{q-r}, 1, 1 + \frac{-1+q}{q-r}, -\frac{ay^{-q+r}}{b}\right]\right)}{b(-1+q)(q-r)}$$

For $q > r = 1$

$$a \equiv a_1$$

$$b \equiv a_q - a_1$$

$$x \to x^2$$

we have

$$y = \frac{1}{\left[1 - \frac{a_q}{a_1} + \frac{a_q}{a_1} e^{(q-1) a_1 x^2}\right]^{\frac{1}{q-1}}}$$

2. Ueber eine Verbesserung der Wien'schen Spectralgleichung[1]

von M. Planck.

(Vorgetragen in der Sitzung vom 19. October 1900.)
(Vgl. oben S. 181.)[1a]

Unter den so aufgestellten Ausdrücken ist mir nun einer besonders aufgefallen, der dem Wien'schen an Einfachheit[11] am nächsten kommt, und der, da letzterer nicht hinreicht, um alle Beobachtungen darzustellen, wohl verdienen würde, daraufhin näher geprüft zu werden. Derselbe ergiebt sich, wenn man setzt (2):

$$\frac{d^2S}{dU^2} = \frac{\alpha}{U(\beta + U)}.^{12} \qquad (*)$$

Er ist bei weitem der einfachste unter allen Ausdrücken, welche S als logarithmische Function von U liefern (was anzunehmen die Wahrscheinlichkeitsrechnung[14] nahe legt) und welche ausserdem für kleine Werte von U in den obigen Wien'schen Ausdruck übergehen. Mit Benutzung der Beziehung

$$\frac{dS}{dU} = \frac{1}{T} \qquad (**)$$

und des Wien'schen „Verschiebungs"gesetzes (1) erhält man hieraus die zweiconstantige[15] Strahlungsformel:[16]

$$E = \frac{C\lambda^{-5}}{e^{c/\lambda T} - 1},$$

$$(*) + (**) \Rightarrow \frac{d}{dU}\left(\frac{1}{T}\right) = \frac{\alpha}{U(\beta + U)}$$

hence $$\frac{dU}{d(1/T)} = \frac{\beta}{\alpha} U + \frac{1}{\alpha} U^2$$

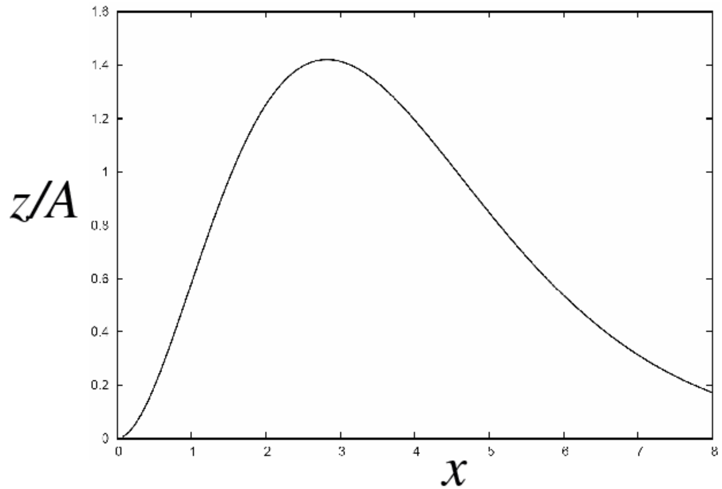
Planck's black-body radiation law can be thought as a $q=2$ to $q=1$ crossover !

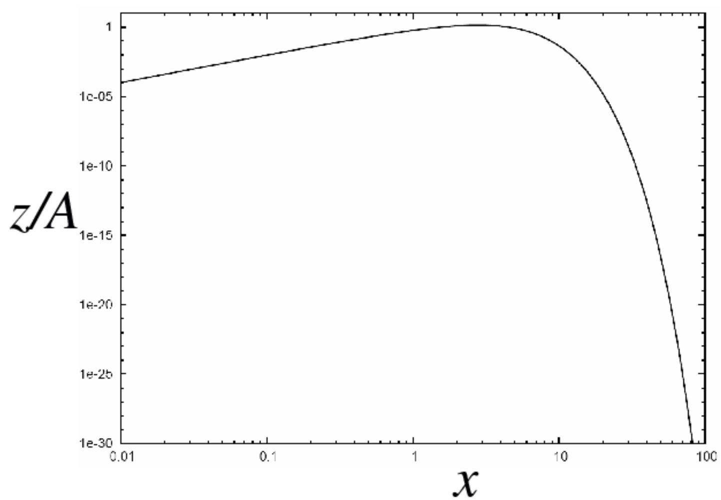
PARTICULAR CASE: PLANCK LAW FOR BLACK-BODY RADIATION!

$$q = 2;\ r = 1;\ \gamma = 2;\ a_1 = 1;\ a_q \to \infty;\ C / a_q \to A > 0$$

$$\Rightarrow y = \frac{A\,x^2}{e^x - 1}$$

$$\Rightarrow z \equiv yx = \frac{A\,x^3}{e^x - 1} \quad (x \equiv h\nu / k_B T)$$





S.M.D. Queiros, C. Anteneodo and C. T., in *Noise and Fluctuations in Econophysics and Finance*, eds. D. Abbott, J.-P. Bouchaud, X. Gabaix and J.L. McCauley, Proc. of SPIE **5848**, 151 (SPIE, Bellingham, WA, 2005)

$$dv = -\gamma \left(v - \frac{\alpha+1}{\beta} \right) dt + \sqrt{2\, v \, \frac{\gamma}{\beta}}\, dW_t,$$

$$P(v) = \frac{1}{Z} \left(\frac{v}{\theta} \right)^{\alpha} \exp_q \left(-\frac{v}{\theta} \right) \qquad e_q^x \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}} \qquad (e_1^x \equiv e^x)$$

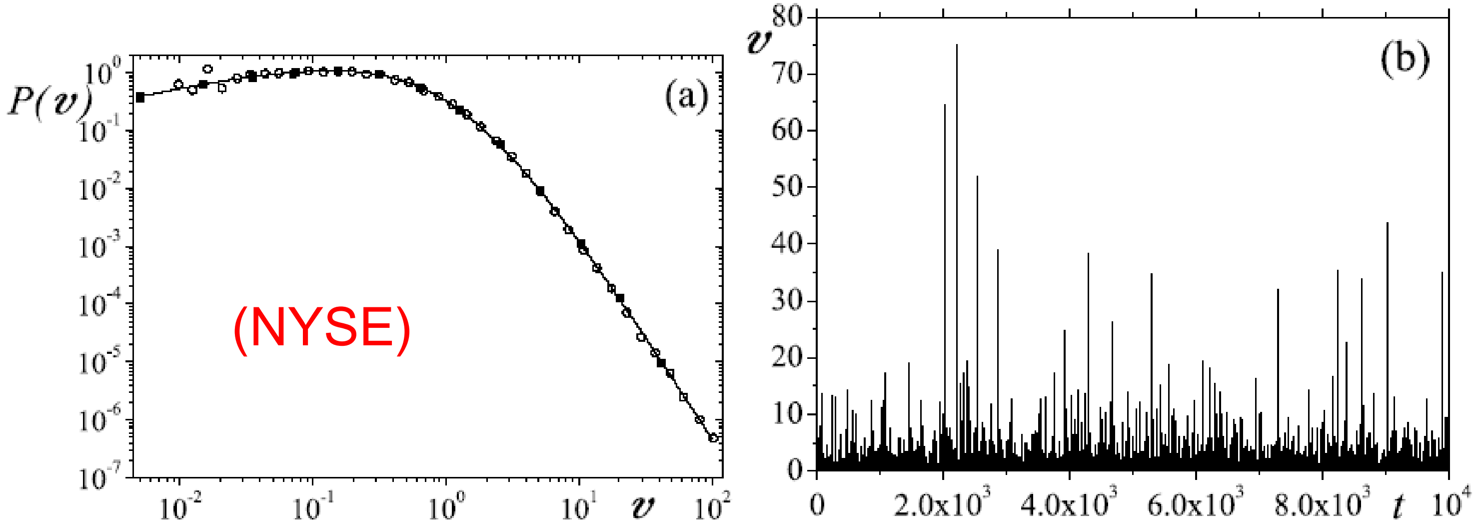
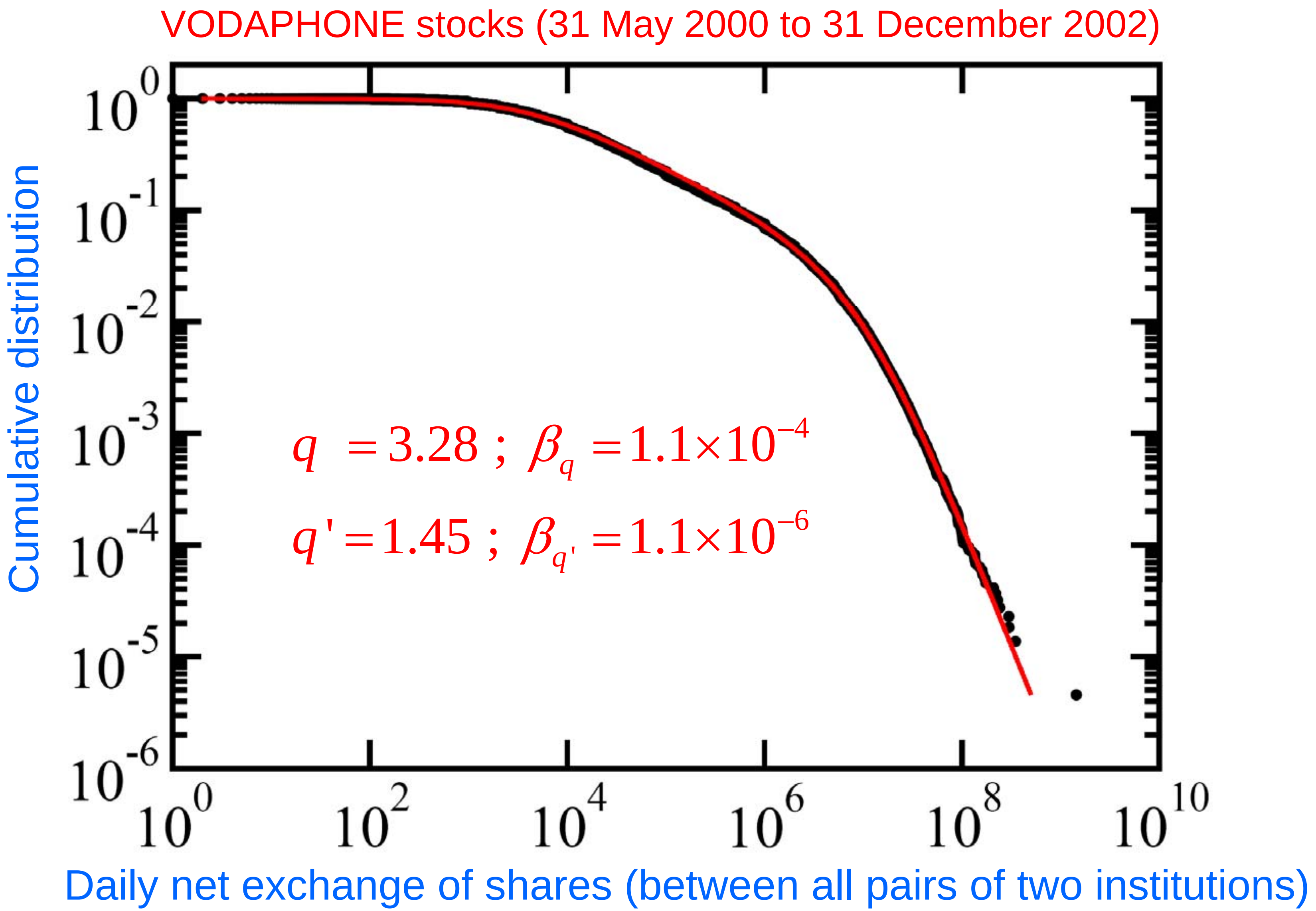


Figure 6. In panel (a) open symbols represents the PDF for the ten-high 1 minute traded volume stocks in NYSE exchange; solid symbols represent the PDF obtained for the numerical realization depicted in panel (b) and line the theoretical PDF Eq. (28). Parameters are $q = 1.17$, $\alpha = 1.79$, $\lambda = 1.42$ and $\delta = 3.09$.

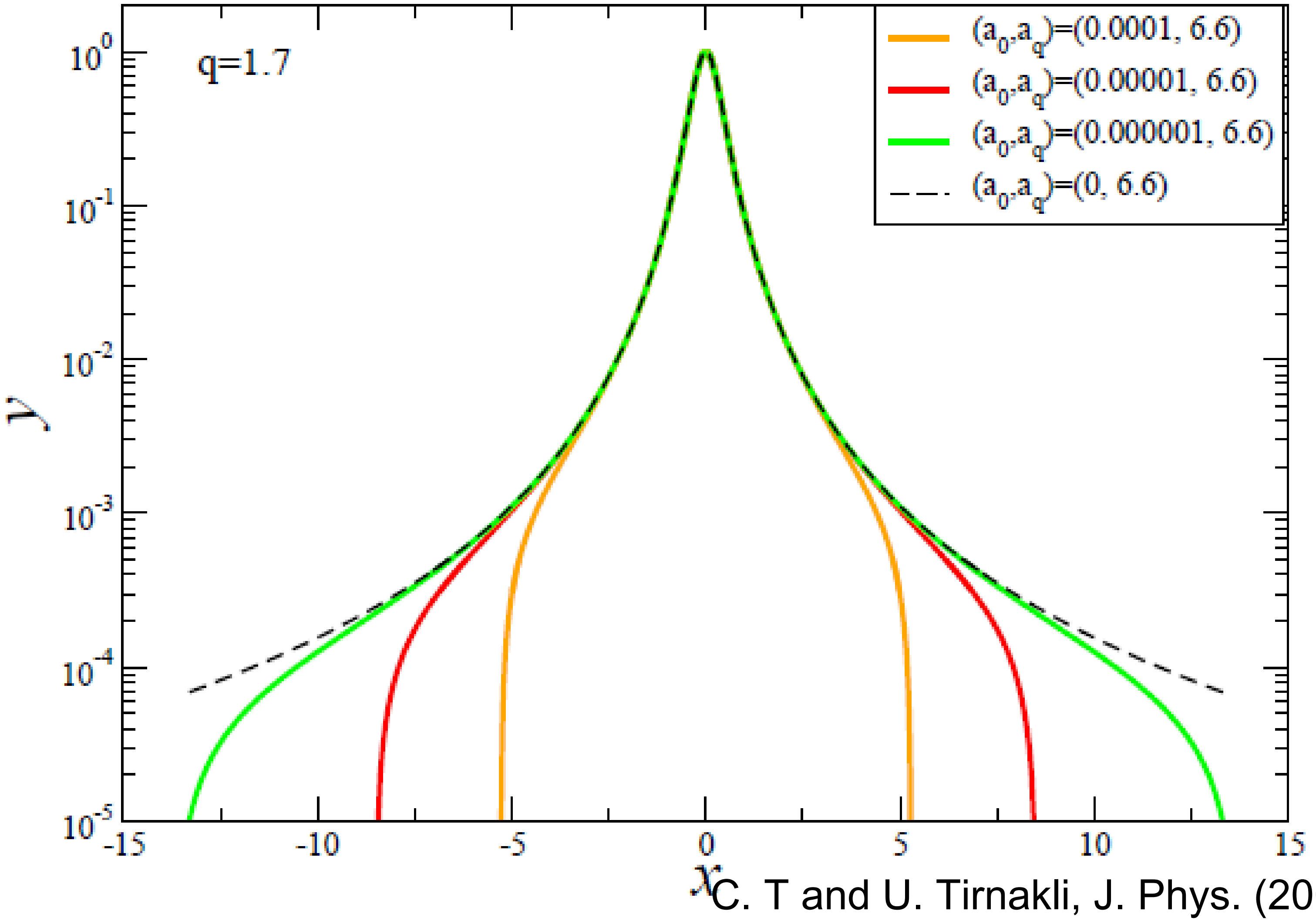
LONDON STOCK EXCHANGE (Block market):

Data: I.I. Zovko; **Fitting**: E.P. Borges (2005)



For $r = 0$ and $q > 1$, the solution is given by

$$x^2 = \frac{1}{a}\left\{ {}_2F_1\left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a}\right] - y\, {}_2F_1\left[\frac{1}{q}, 1, 1 + \frac{1}{q}, -\frac{b}{a}\, y^q\right]\right\}$$



$a \equiv a_0$

$b \equiv a_q - a_0$

$x \to x^2$

C. T and U. Tirnakli, J. Phys. (2009) in press

PHYSICAL REVIEW E **79**, 056209 (2009)

Closer look at time averages of the logistic map at the edge of chaos

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Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey

Constantino Tsallis
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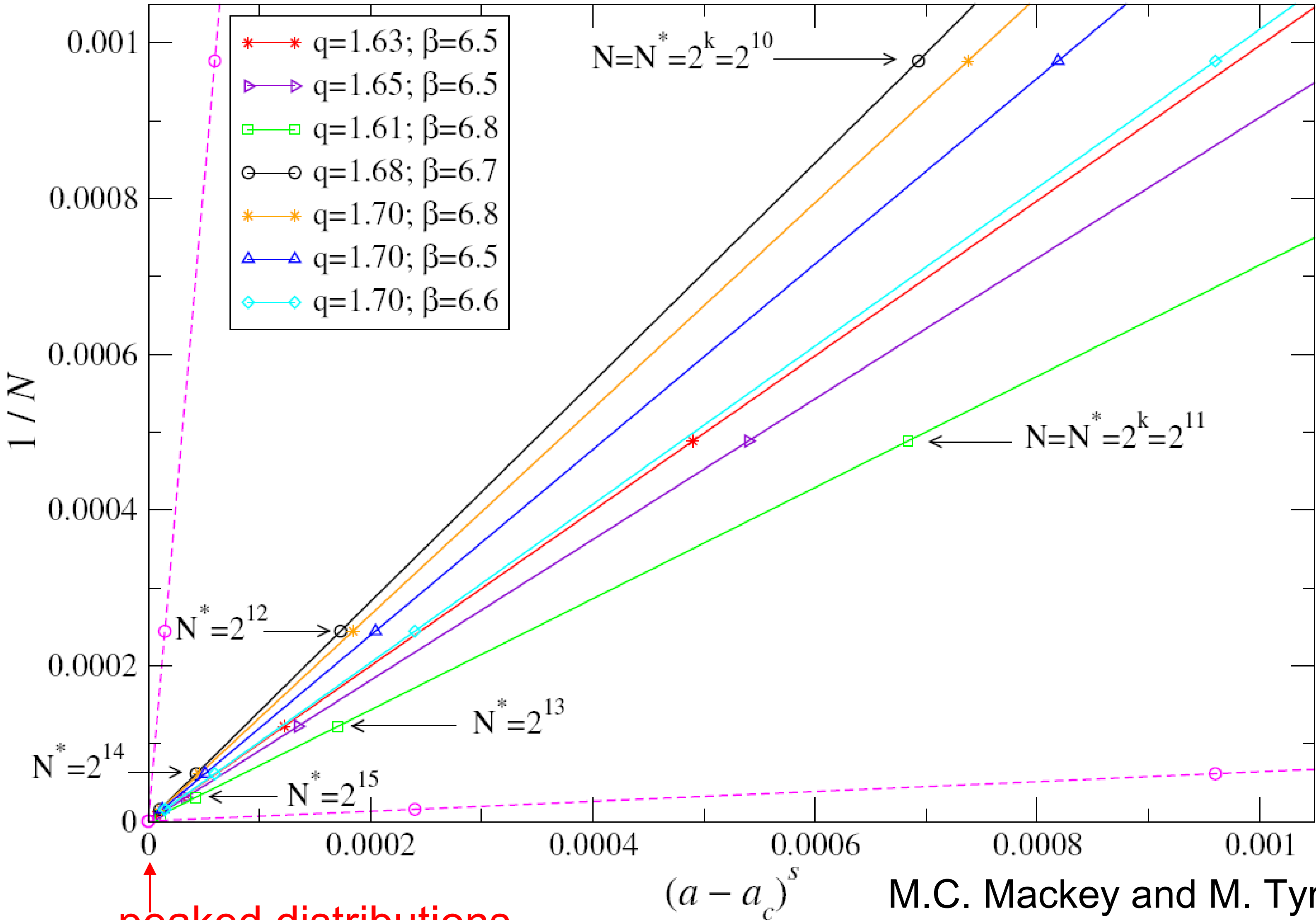
Christian Beck
School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom
(Received 8 February 2008; revised manuscript received 19 December 2008; published 11 May 2009)

See also
U. Tirnakli, C. Beck and C. T., Phys. Rev. E **75**, 040106(R) (2007)



Gaussians

peaked distributions

M.C. Mackey and M. Tyran-Kaminska
Phys Rep **422** (2006) 167

A. Robledo and M.A. Fuentes

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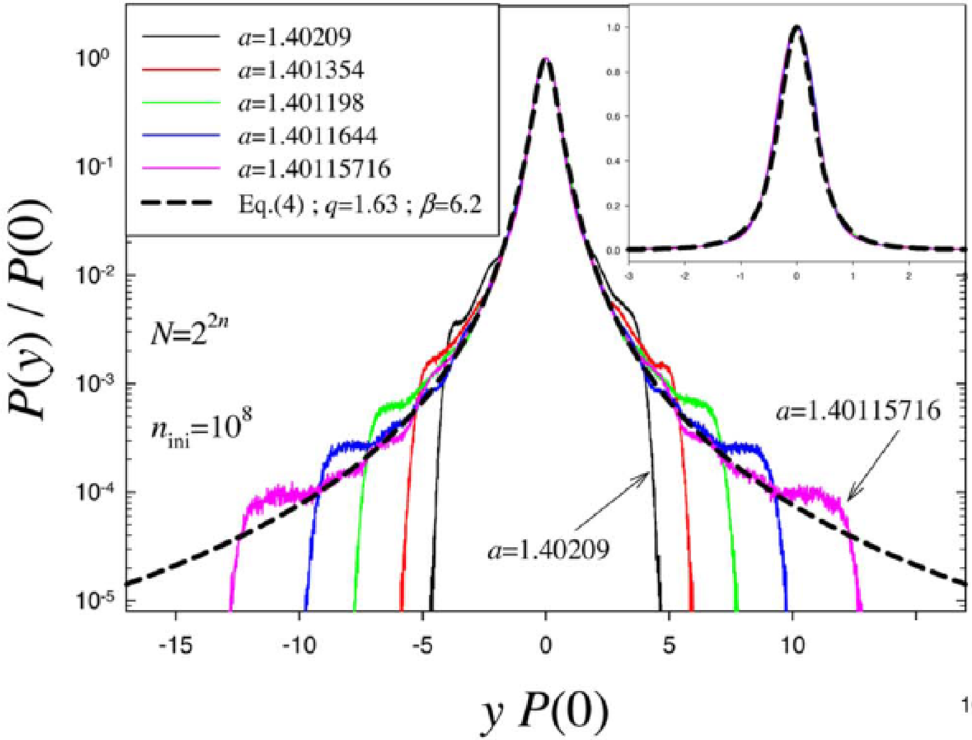
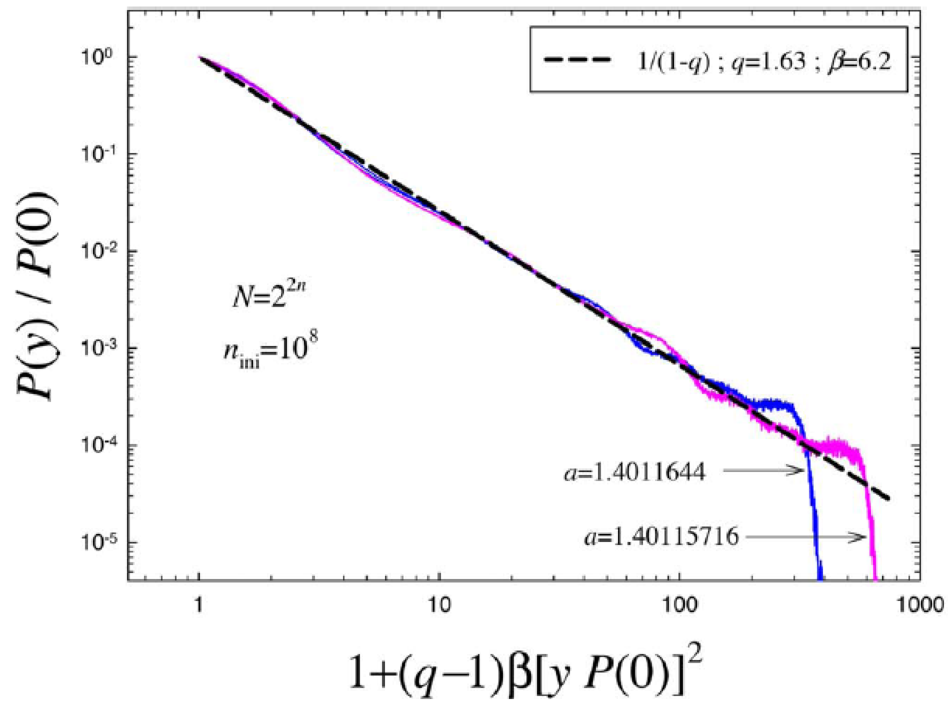
[see also U. Tirnakli, C. T. and C. Beck, 0906.1262 cond-mat.stat-mech]



U. Tirnakli, C. T. and C. Beck
Phys Rev E **79** (2009) 056209

$q = 1.63$

$\beta = 6.2$

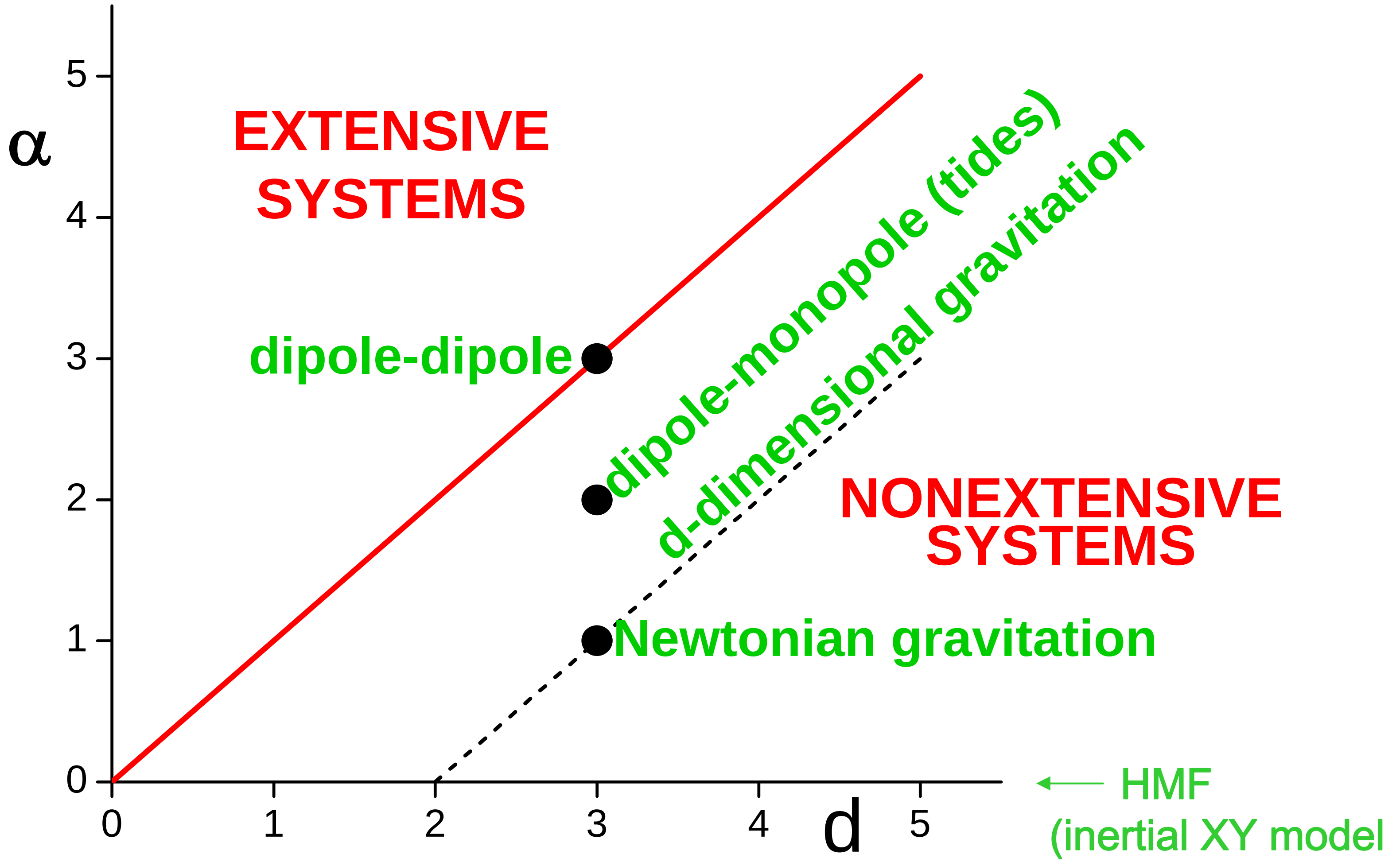


CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

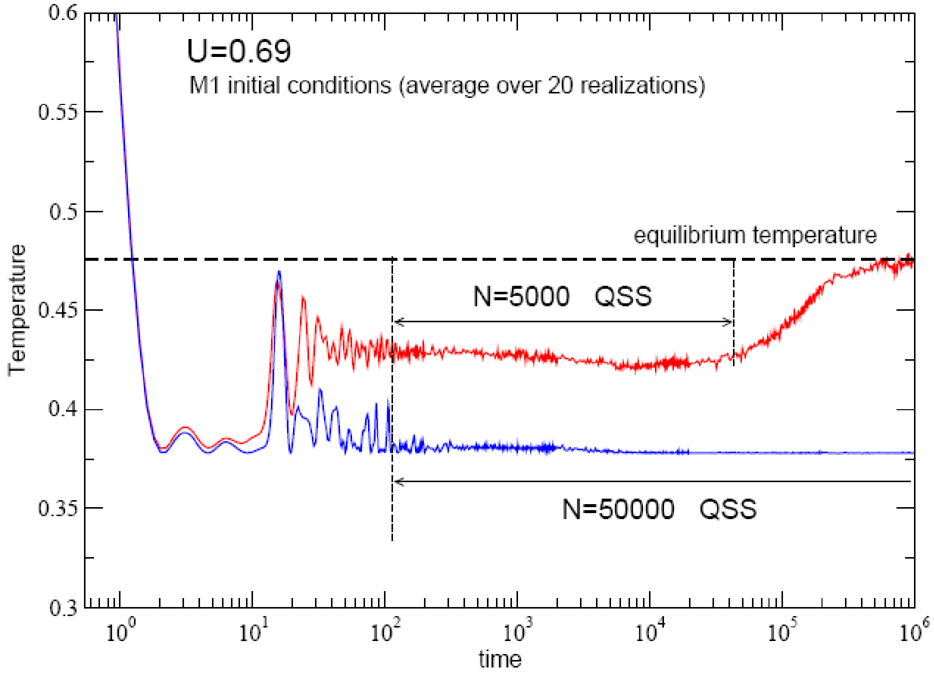
$$V(\vec{r}) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty) \qquad (A > 0, \quad \alpha \geq 0)$$

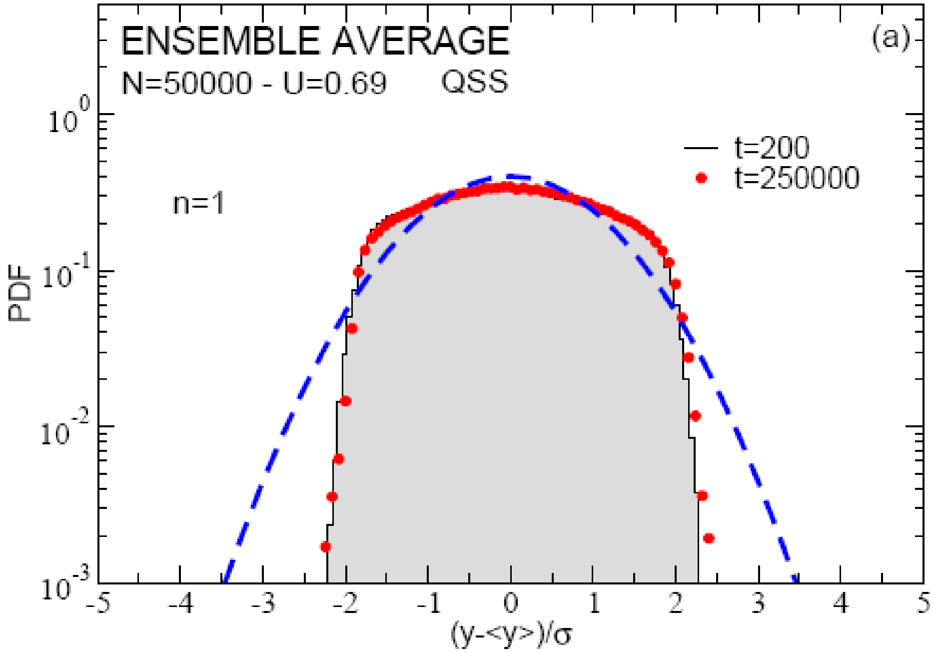
integrable if $\alpha / d > 1$ *(short-ranged)*

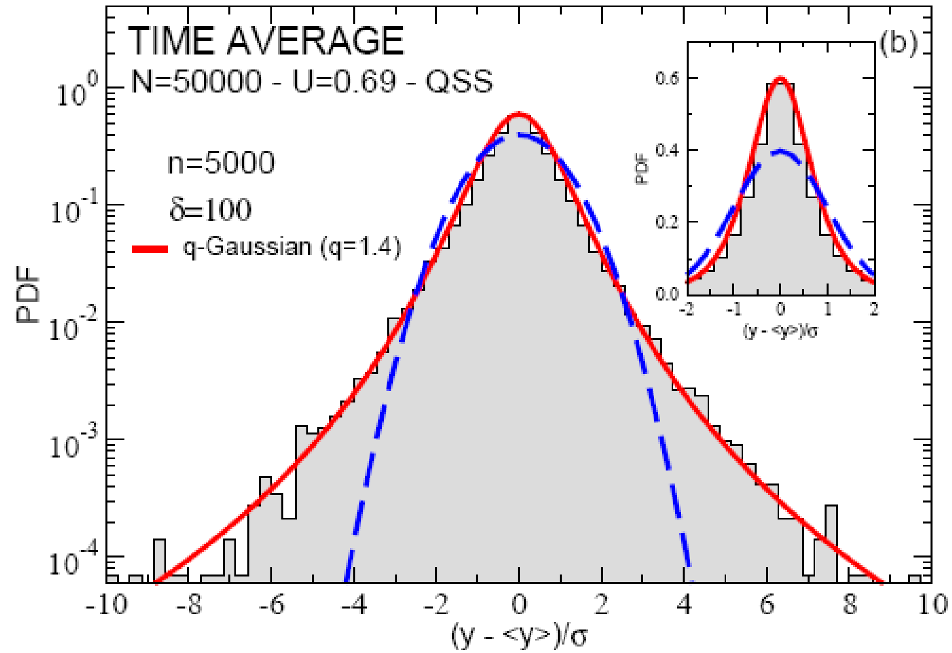
non-integrable if $0 \leq \alpha / d \leq 1$ *(long-ranged)*



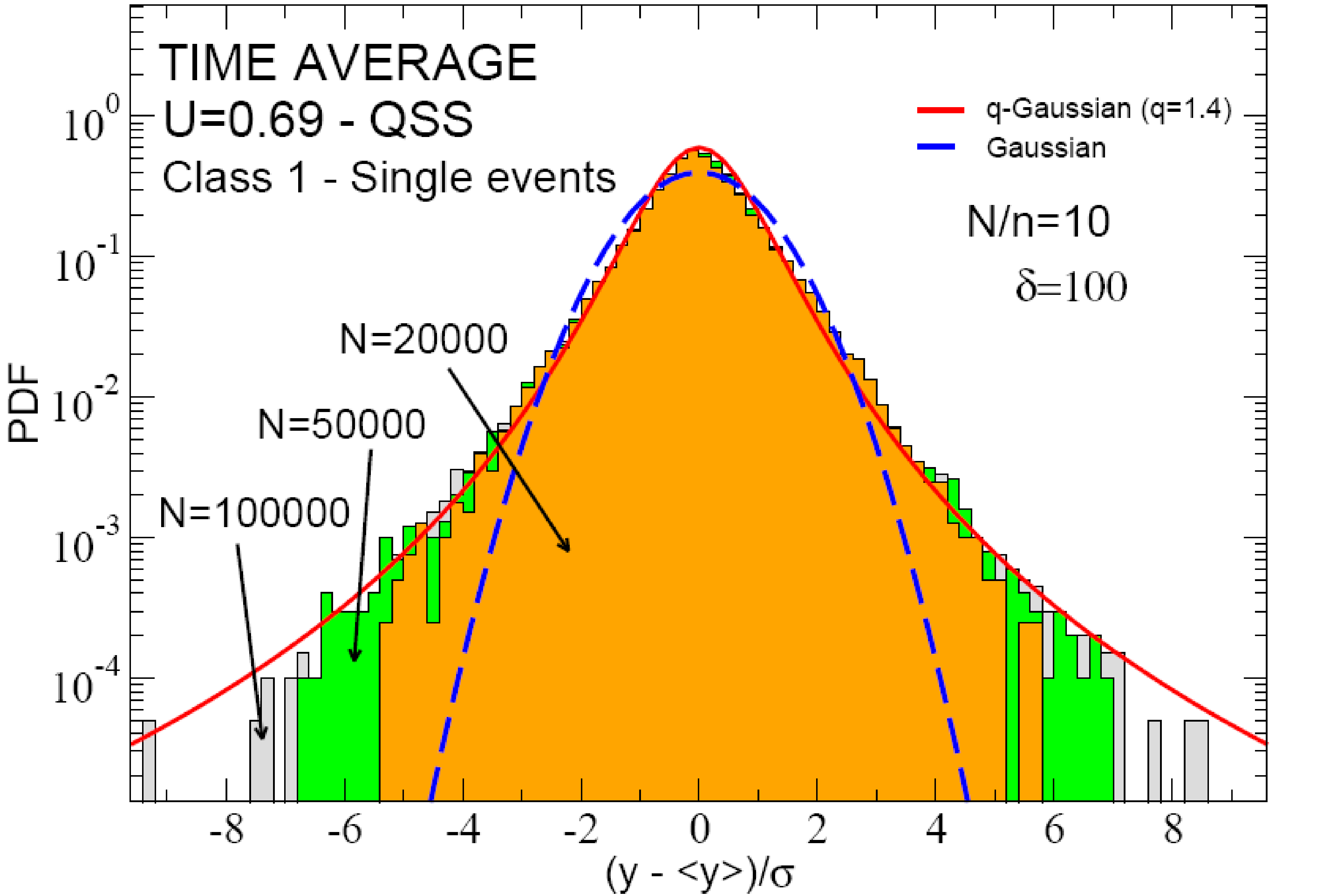
HMF MODEL







A. Pluchino, A. Rapisarda and C. T., Europhys Lett **80**, 26002 (2007)

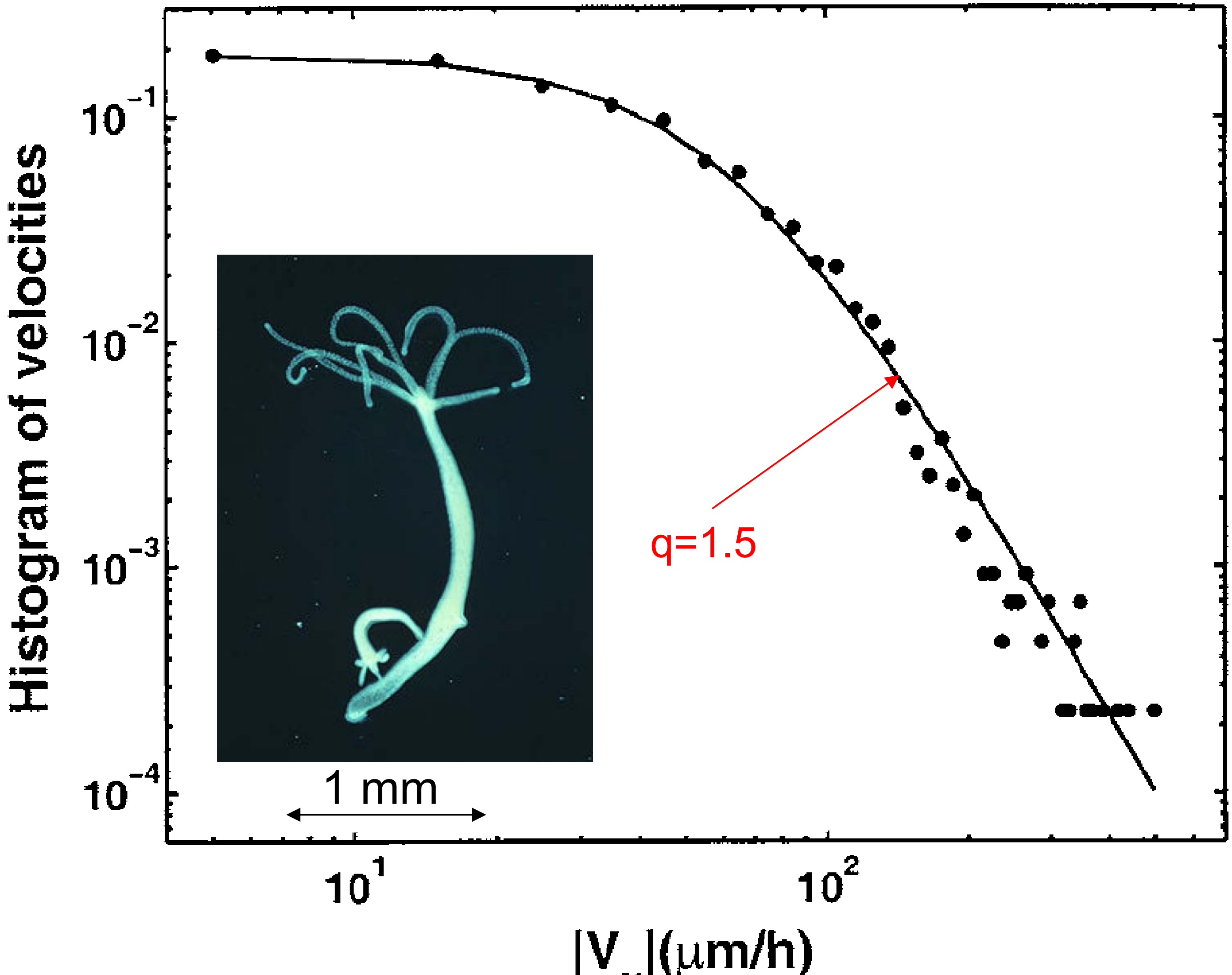


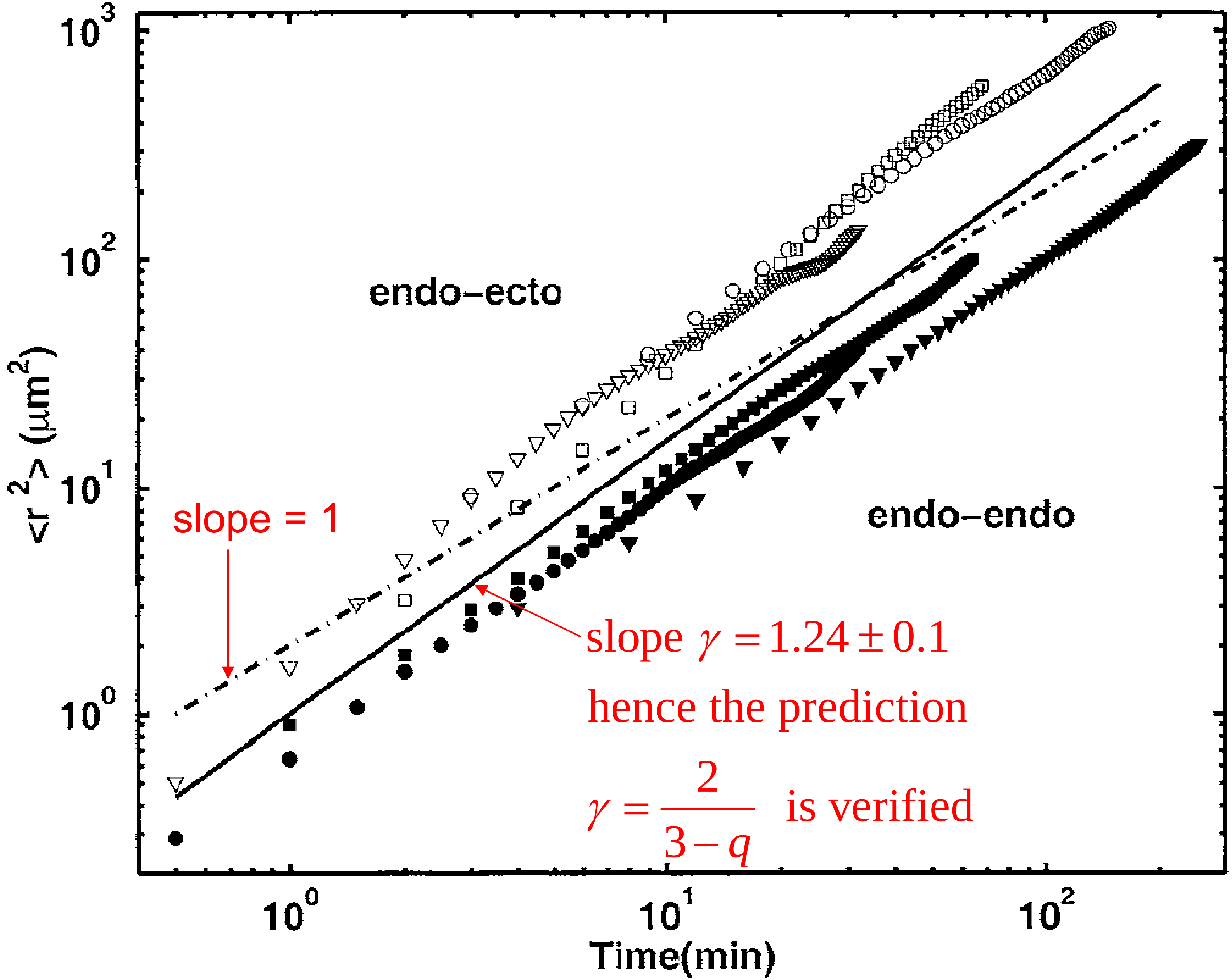
[See A. Pluchino, A. Rapisarda and C. T., Europhys Lett **85**, 60006 (2009)]

SOME EXPERIMENTAL, OBSERVATIONAL AND COMPUTATIONAL VERIFICATIONS AND APPLICATIONS

Hydra viridissima:

A Upadhyaya, J-P Rieu, JA Glazier and Y Sawada, Physica A **293**, 549 (2001)

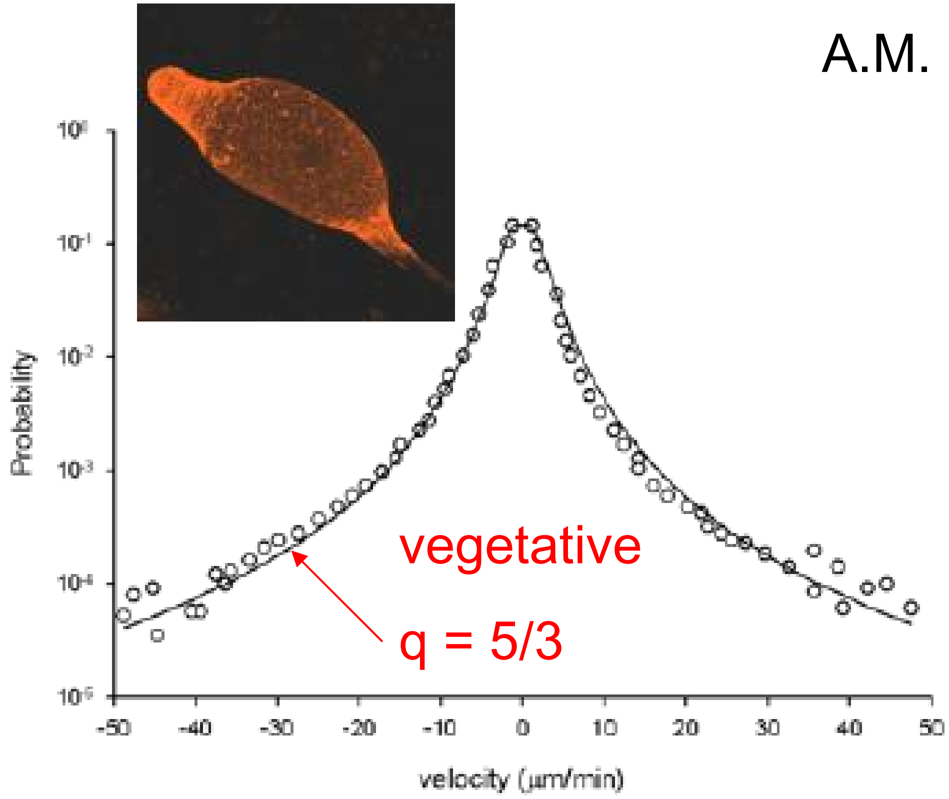


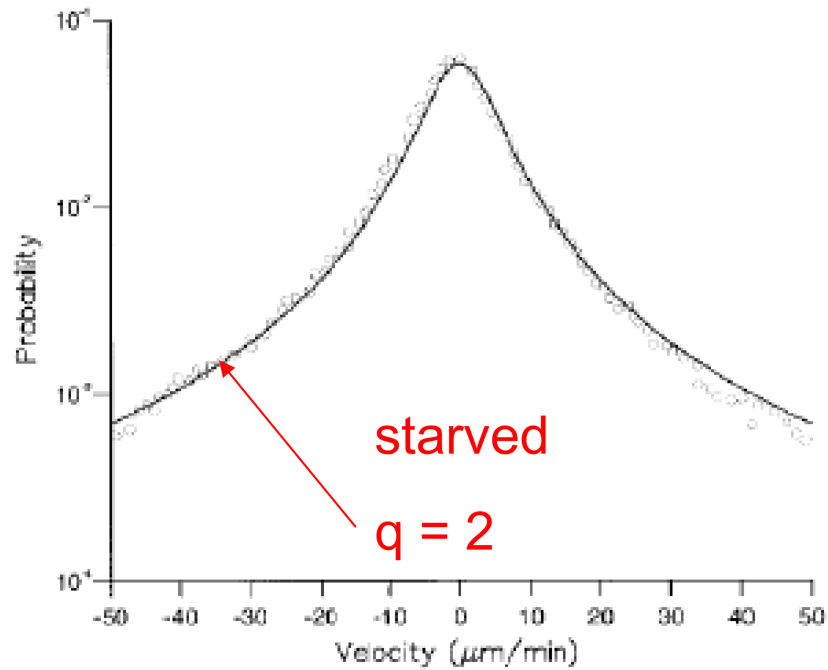


A Upadhyaya, J-P Rieu, JA Glazier and Y Sawada, Physica A **293**, 549 (2001)

Dictyostelium discoideum (cells):

A.M. Reynolds, Physica A **389**, 273 (2010)





$$P(v) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\alpha}{2}\right)} \frac{v_a^{\alpha}}{\left[v_a^2+v^2\right]^{\frac{\alpha+1}{2}}} \equiv \frac{P(0)}{\left[1+(q-1)\;\beta\;v^2\right]^{\frac{1}{q-1}}}$$

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

RAPID COMMU

PHYSICAL REVIEW A **67**, 051402(R) (2003)

Anomalous diffusion and Tsallis statistics in an optical lattice

Eric Lutz

Sloane Physics Laboratory, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120

(Received 26 February 2003; published 27 May 2003)

We point out a connection between anomalous transport in an optical lattice and Tsallis' generalized statistics. Specifically, we show that the momentum equation for the semiclassical Wigner function which describes atomic motion in the optical potential, belongs to a class of transport equations recently studied by Borland [Phys. Lett. A **245**, 67 (1998)]. The important property of these ordinary linear Fokker-Planck equations is that their stationary solutions are exactly given by Tsallis distributions. An analytical expression of the Tsallis index q in terms of the microscopic parameters of the quantum-optical problem is given and the spatial coherence of the atomic wave packets is discussed.

(i) The distribution of atomic velocities is a q-Gaussian;

(ii) $q = 1 + \frac{44E_R}{U_0}$ where $E_R \equiv \text{recoil energy}$

$U_0 \equiv \text{potential depth}$

PRL 96, 110601 (2006) PHYSICAL REVIEW LETTERS week ending 24 MARCH 2006

Tunable Tsallis Distributions in Dissipative Optical Lattices

P. Douglas, S. Bergamini, and F. Renzoni

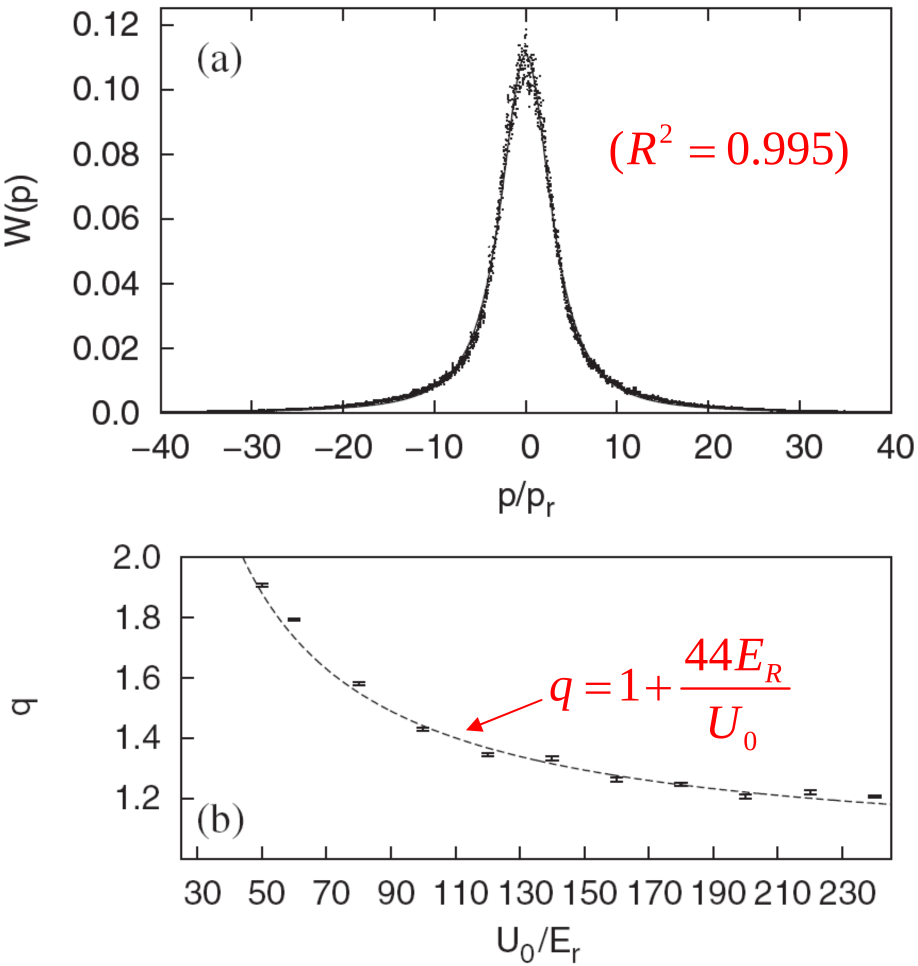
Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom

(Received 10 January 2006; published 24 March 2006)

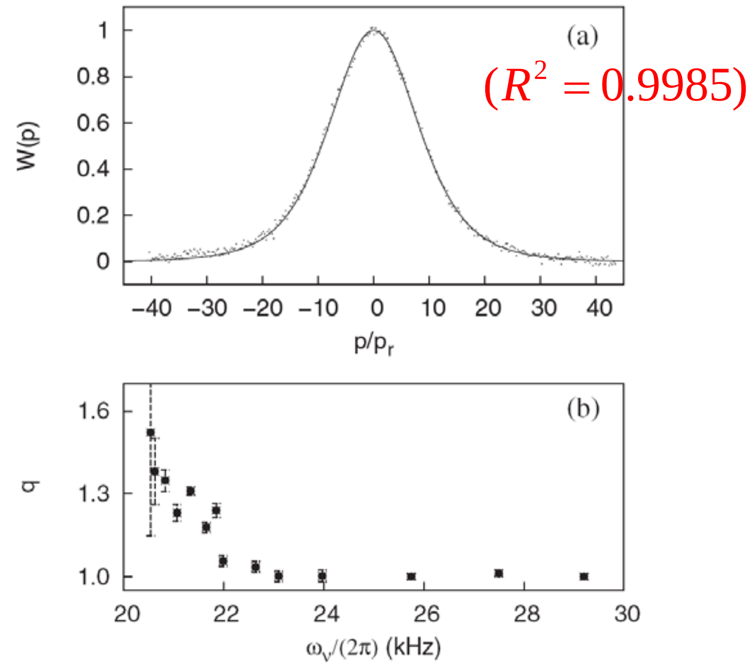
We demonstrated experimentally that the momentum distribution of cold atoms in dissipative optical lattices is a Tsallis distribution. The parameters of the distribution can be continuously varied by changing the parameters of the optical potential. In particular, by changing the depth of the optical lattice, it is possible to change the momentum distribution from Gaussian, at deep potentials, to a power-law tail distribution at shallow optical potentials.

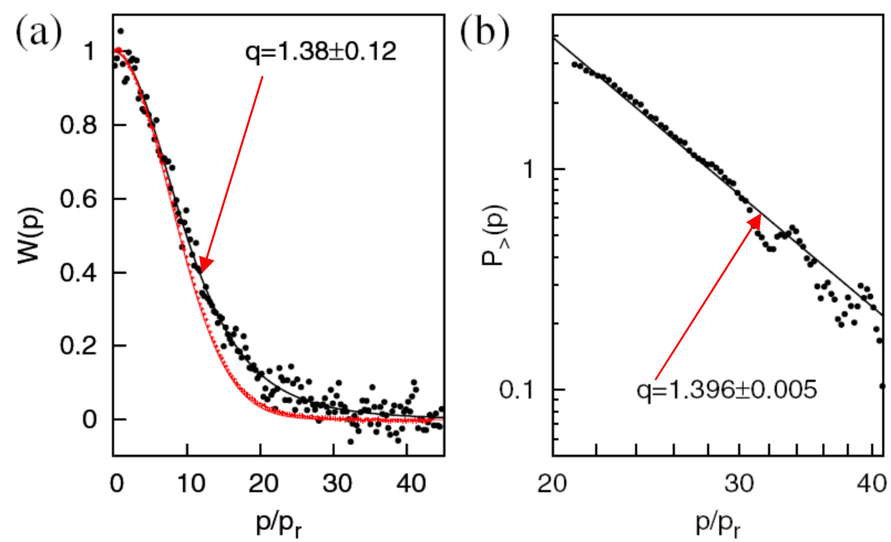
Experimental and computational verifications

by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)



(Computational verification: quantum Monte Carlo simulations)





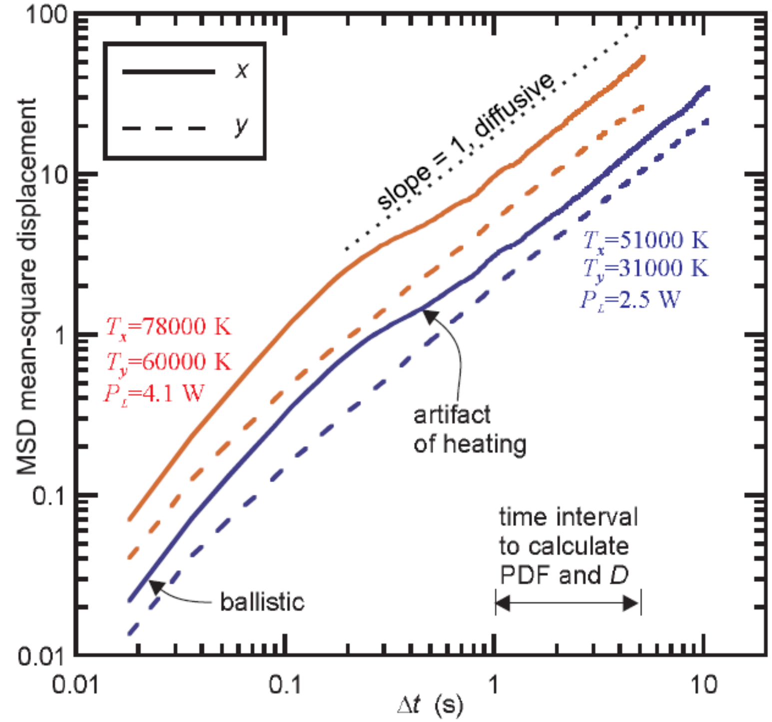
(Experimental verification: Cs atoms)

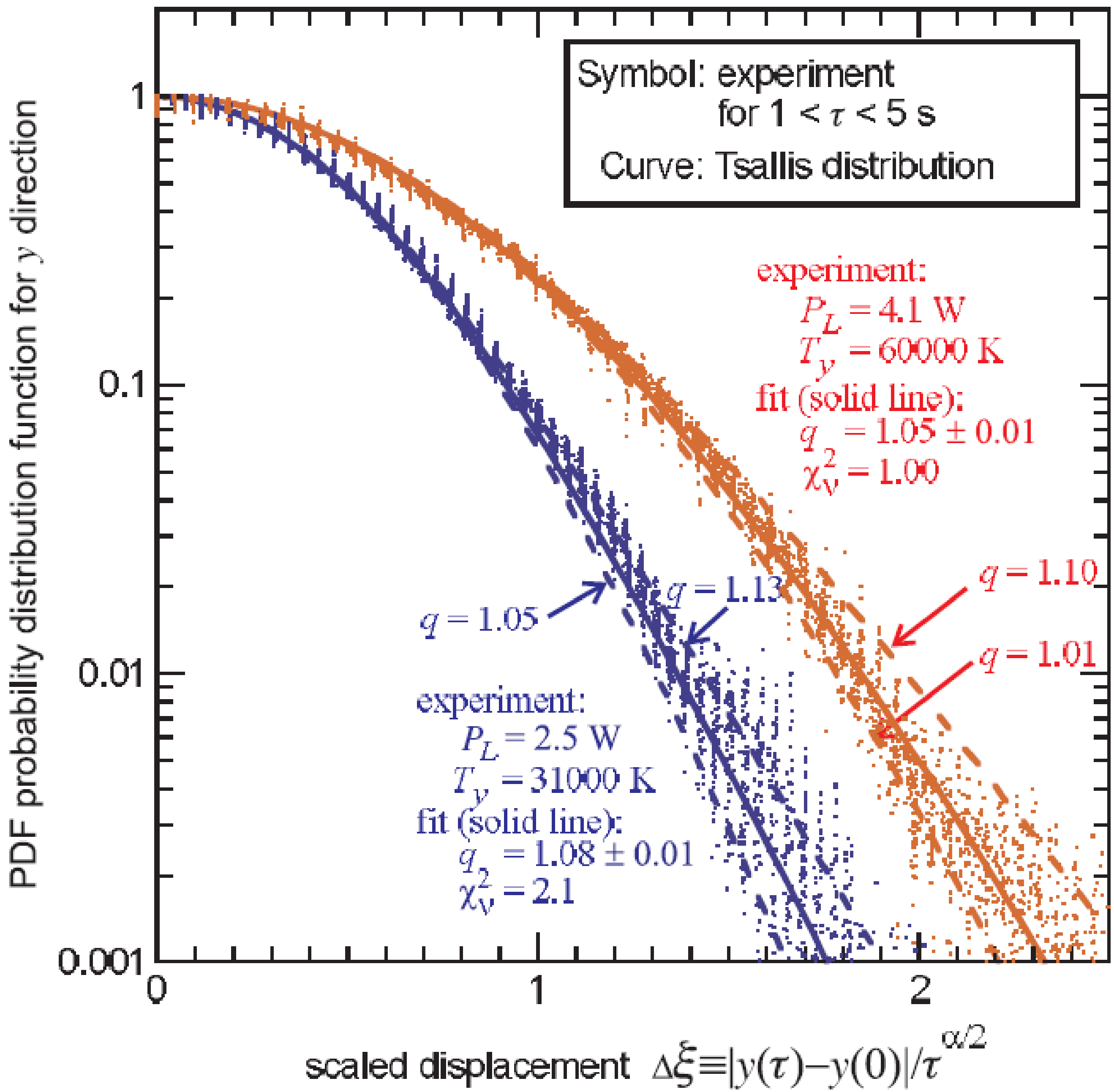
Superdiffusion and Non-Gaussian Statistics in a Driven-Dissipative 2D Dusty Plasma

Bin Liu and J. Goree
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA
(Received 1 June 2007; published 6 February 2008)

Anomalous diffusion and non-Gaussian statistics are detected experimentally in a two-dimensional driven-dissipative system. A single-layer dusty plasma suspension with a Yukawa interaction and frictional dissipation is heated with laser radiation pressure to yield a structure with liquid ordering. Analyzing the time series for mean-square displacement, superdiffusion is detected at a low but statistically significant level over a wide range of temperatures. The probability distribution function fits a Tsallis distribution, yielding q, a measure of nonextensivity for non-Gaussian statistics.

$$< r^2 > \;\propto\; t^\alpha$$





B. Liu and J. Goree, Phys Rev Lett **100**, 055003 (2008)

LASER COOLING:

PRL **102,** 063001 (2009) PHYSICAL REVIEW LETTERS week ending 13 FEBRUARY 2009

Power-Law Distributions for a Trapped Ion Interacting with a Classical Buffer Gas

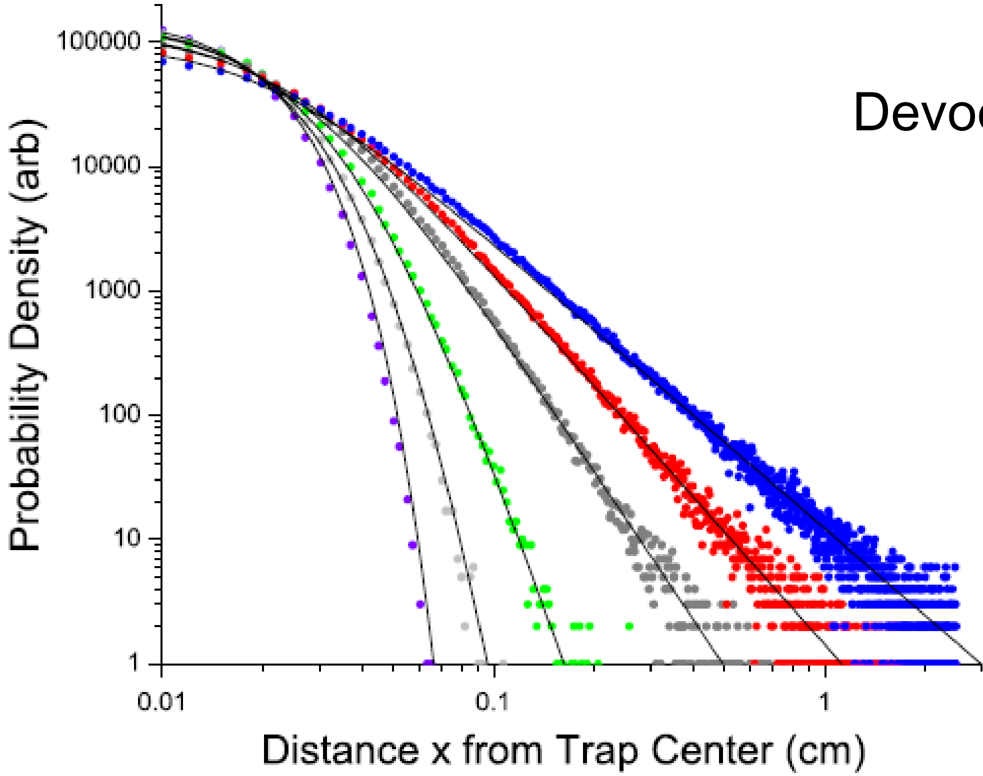
Ralph G. DeVoe

Physics Department, Stanford University, Stanford, California 94305, USA

(Received 3 November 2008; published 10 February 2009)

Classical collisions with an ideal gas generate non-Maxwellian distribution functions for a single ion in a radio frequency ion trap. The distributions have power-law tails whose exponent depends on the ratio of buffer gas to ion mass. This provides a statistical explanation for the previously observed transition from cooling to heating. Monte Carlo results approximate a Tsallis distribution over a wide range of parameters and have *ab initio* agreement with experiment.

Devoe, Phys Rev Lett **102** (2009) 063001



$$T(x) = \frac{T(0)}{\left[1 + (q-1)\left(\frac{x}{\sigma}\right)^2\right]^{\frac{1}{q-1}}}$$

FIG. 1 (color online). Monte Carlo distributions for a single $^{136}Ba^+$ ion cooled by six different buffer gases at 300 K ranging from $m_B = 4$ (left) to $m_B = 200$ (right). Note the evolution from Gaussian to power law (straight line) as the mass increases. The solid lines are Tsallis functions [Eq. (7)] with fixed $\sigma = 0.0185$ cm and the exponents of Table I.

TABLE I. Tsallis parameters n and q_T fit from Fig. 1.

Buffer gas	m_I/m_B	n	q_T
He	34.5	>60	1.03
Ar	3.40	8.2	1.12
Kr	1.70	3.8	1.26
Xe	1.0	1.98	1.51
170	0.80	1.50	1.80
200	0.68	1.15	1.87

SPIN RELAXATION IN SPIN GLASSES (NEUTRON SPIN ECHO):

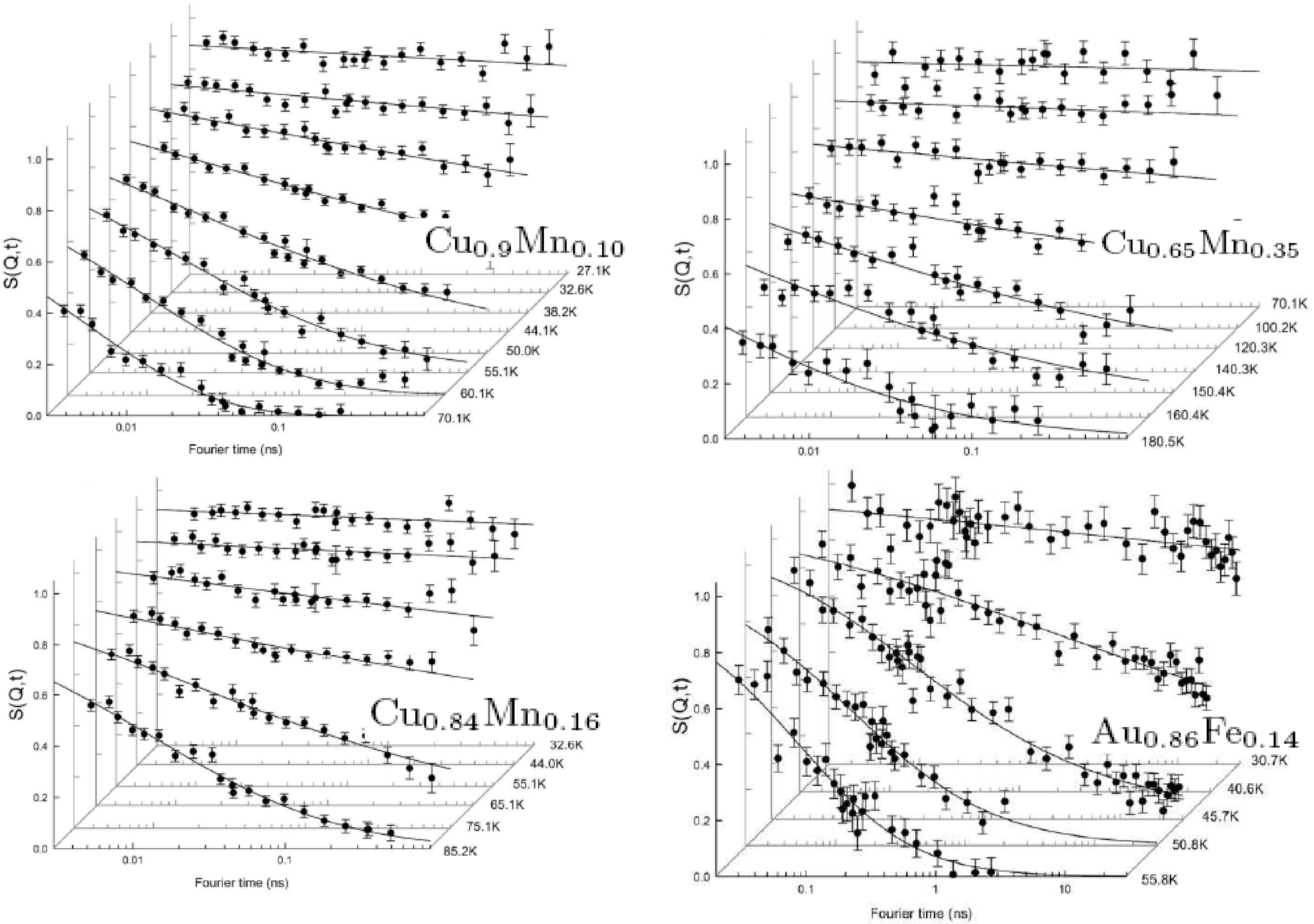
PRL 102, 097202 (2009) PHYSICAL REVIEW LETTERS week ending 6 MARCH 2009

Generalized Spin-Glass Relaxation

R. M. Pickup,[1] R. Cywinski,[2,*] C. Pappas,[3] B. Farago,[4] and P. Fouquet[4]

[1]*School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom*
[2]*School of Applied Sciences, University of Huddersfield, Huddersfield HD1 3DH, United Kingdom*
[3]*Helmholtz Center Berlin for Materials and Energy, Glienickerstrasse 100, 14109, Berlin, Germany*
[4]*Institut Laue Langevin, 6 rue Jules Horowitz, 38000 Grenoble, France*
(Received 18 July 2008; published 4 March 2009)

Spin relaxation close to the glass temperature of CuMn and AuFe spin glasses is shown, by neutron spin echo, to follow a generalized exponential function which explicitly introduces hierarchically constrained dynamics and macroscopic interactions. The interaction parameter is directly related to the normalized Tsallis nonextensive entropy parameter q and exhibits universal scaling with reduced temperature. At the glass temperature $q = 5/3$ corresponding, within Tsallis' q statistics, to a mathematically defined critical value for the onset of strong disorder and nonlinear dynamics.



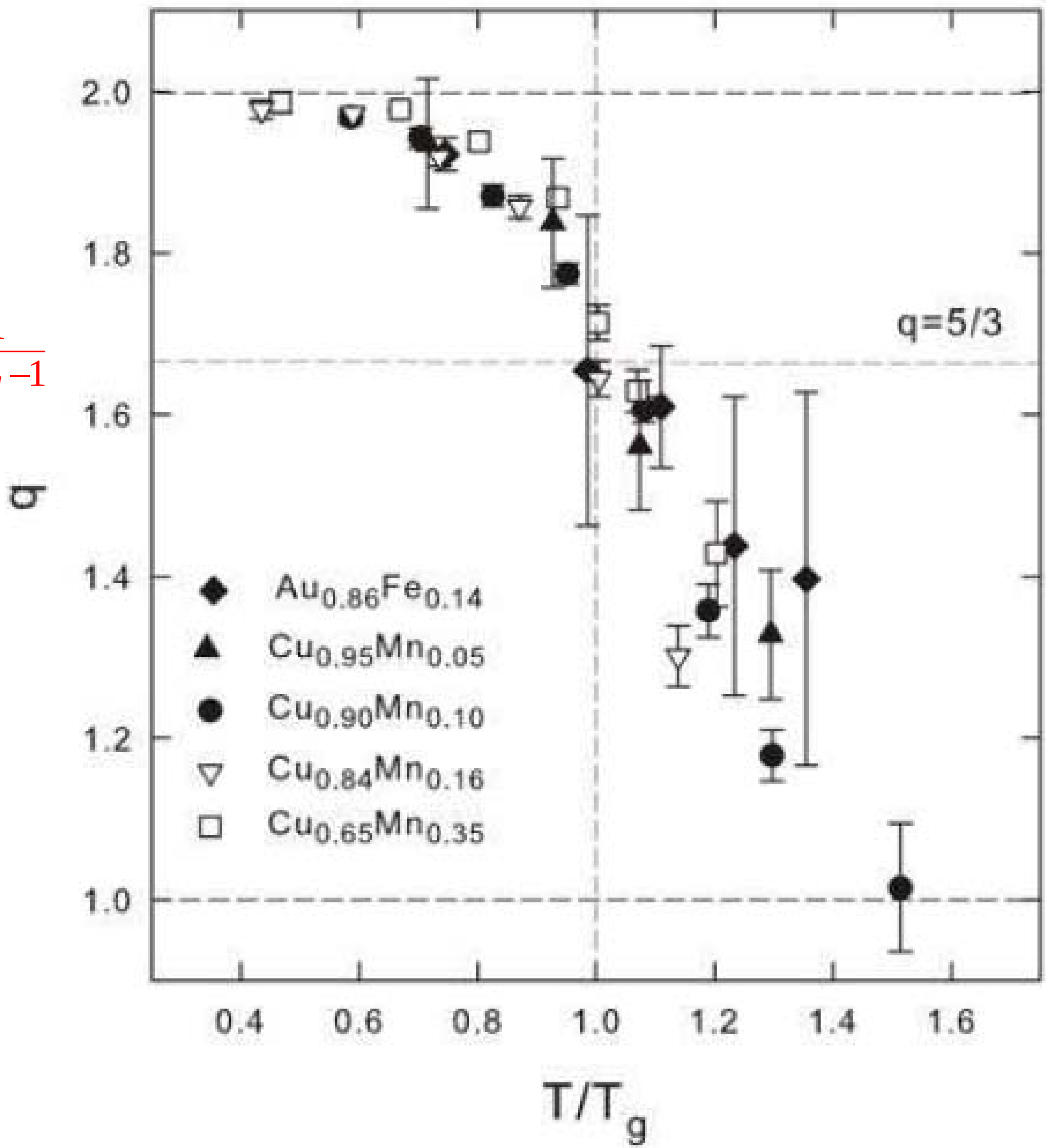
Pickup, Cywinski, Pappas, Farago, Fouquet, Phys Rev Lett **102** (2009) 097202

SPIN RELAXATION IN SPIN GLASSES (NEUTRON SPIN ECHO):

$$\phi(t) = \left[1 + \frac{q-1}{2-q}\left(\frac{t}{\tau}\right)^{\beta}\right]^{-\frac{2-q}{q-1}}$$

$$\equiv \left[1 + (q_{rel} - 1)\left(\frac{t}{\tau}\right)^{\beta}\right]^{-\frac{1}{q_{rel}-1}}$$

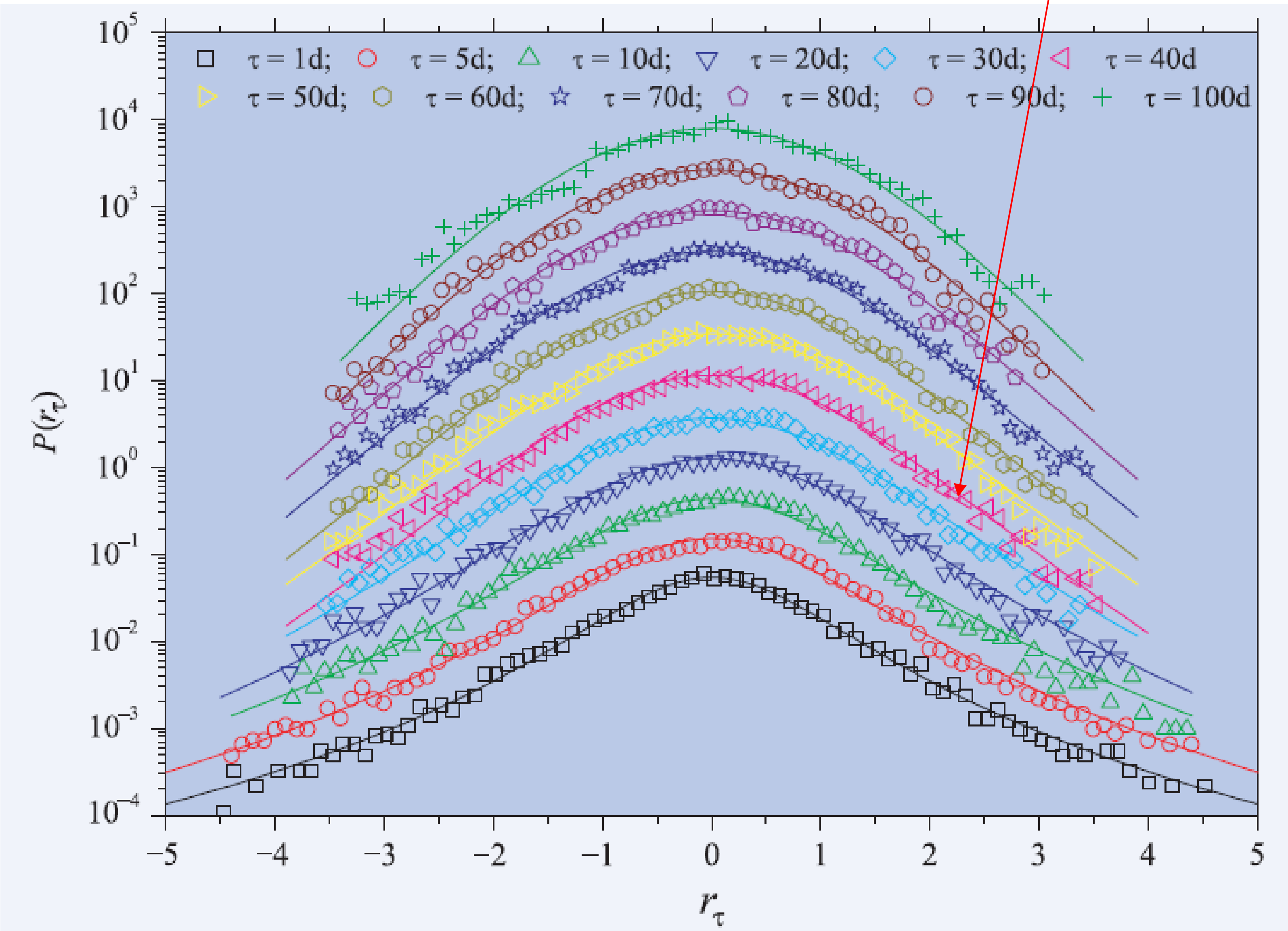
$$q_{rel} \equiv \frac{1}{2-q}$$



Pickup, Cywinski, Pappas, Farago, Fouquet, Phys Rev Lett **102** (2009) 097202

RETURNS: New York Stock Exchange

q-Gaussians



τ	q
1	1.488
5	1.419
10	1.369
20	1.287
30	1.233
40	1.192
50	1.175
60	1.151
70	1.135
80	1.114
90	1.085
100	1.073

S.M.D. Queiros, Quant. Finance **5**, 476 (2005)

The Astrophysical Journal, 703:311–324, 2009 September 20 doi:10.1088/0004-637X/703

COMPRESSIBLE "TURBULENCE" OBSERVED IN THE HELIOSHEATH BY *VOYAGER 2*

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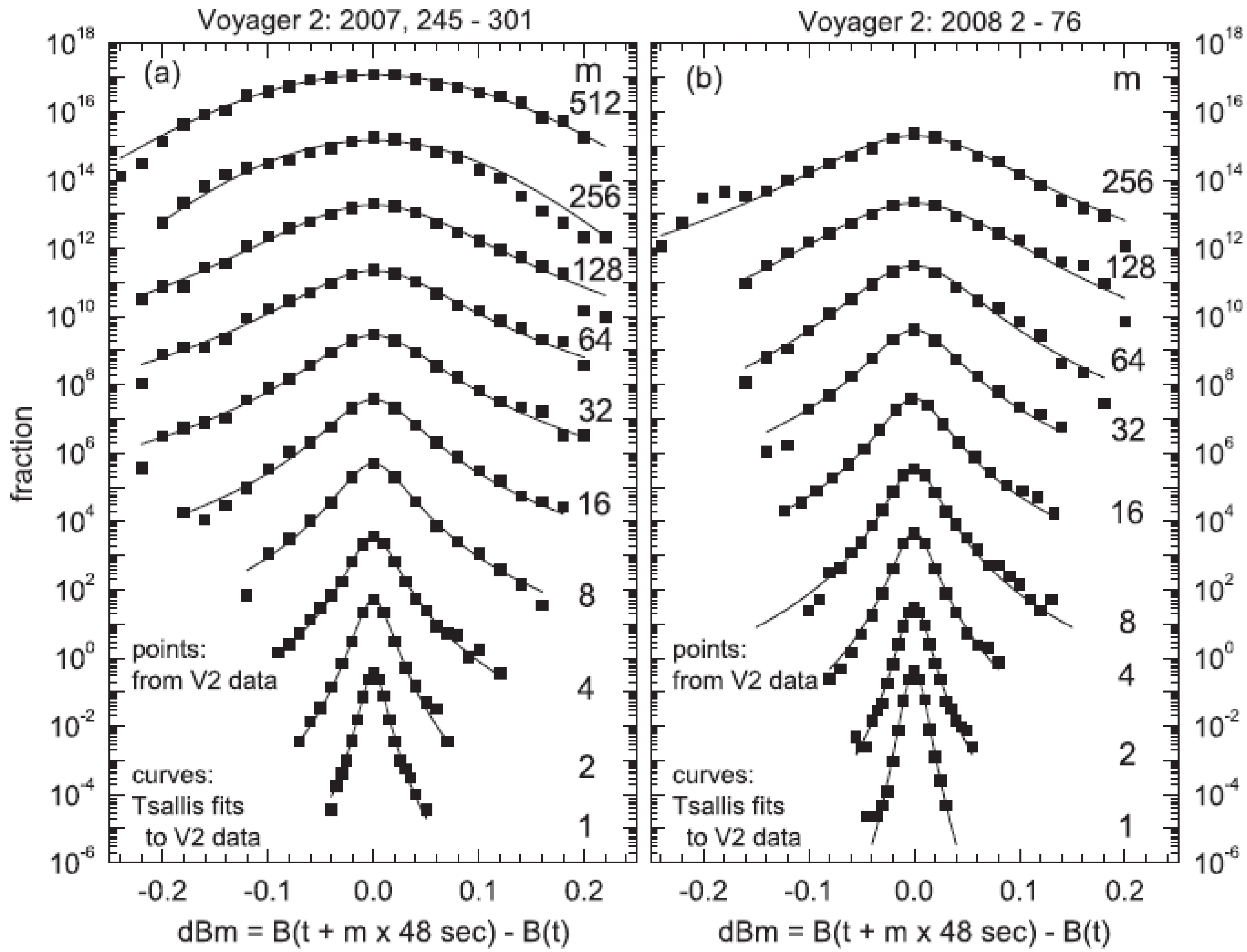
[1] Geospace Physics Laboratory, Code 673, NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA; Leonard.F.Burlaga@NASA.gov
[2] Institute for Astrophysics and Computational Sciences, Catholic University of America, Washington DC 20064, USA; nfnudel@yahoo.com
Received 2009 June 2; accepted 2009 July 22; published 2009 August 27



Introduction to Nonextensive Statistical Mechanics

APPROACHING A COMPLEX WORLD

Constantino Tsallis

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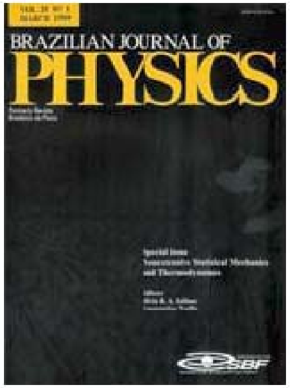
BOOKS ON NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS



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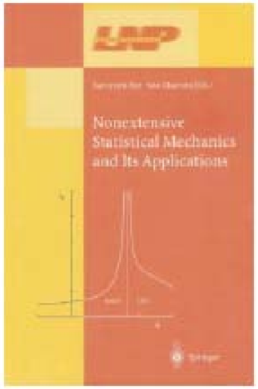

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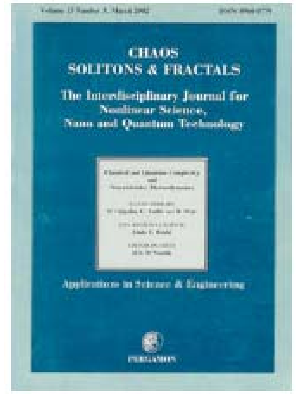

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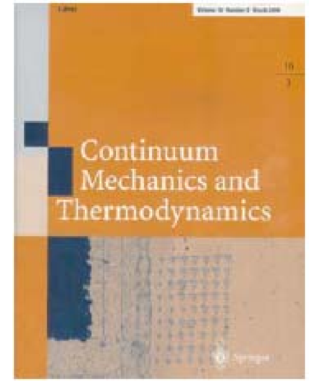

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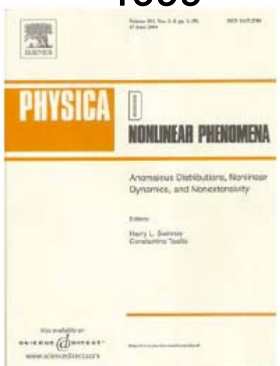

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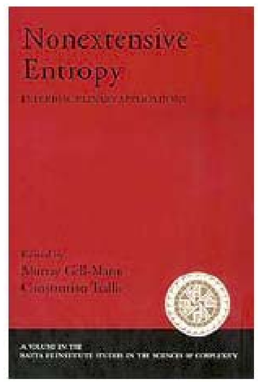

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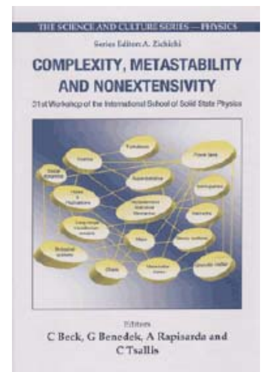

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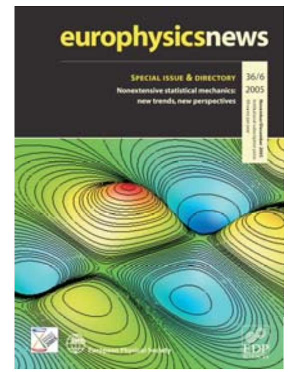

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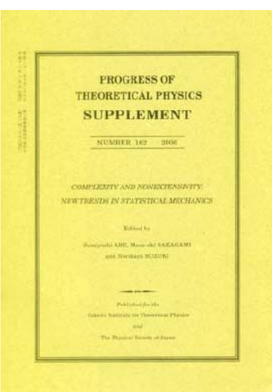

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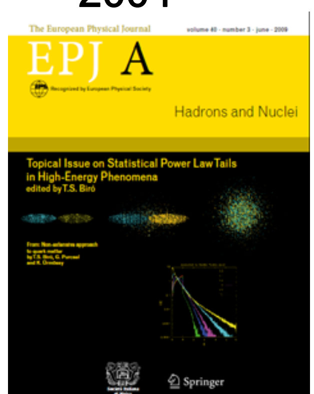

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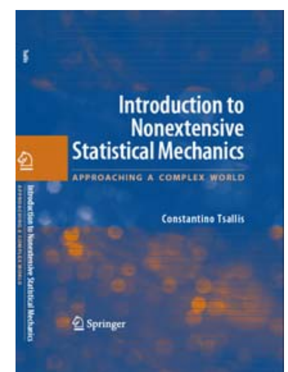

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[24 October 2009]