Risk Aggregation, Dependence Structure and Diversification Benefits

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Acknowledgement

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- Roland Bürgi who programmed the whole analysis and developed a method to generate hierarchical multivariate distributions with copulas
- Roger Iles
- And Michel Dacorogna

It appeared as a chapter in the Incisive Media Riskbooks: "*Stress testing for financial institutions*", 2008, edited by D. Rösch and H. Scheule, chapter 12, pages 265-305.

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Agenda

Determining Risk-Adjusted Capital

- The Risk-Adjusted Capital (RAC) of an insurance company is evaluated on the basis of a quantitative model of its different risks
- We first need to identify the various sources of risk. One usually distinguishes four large risk categories:
  1. Underwriting risk (or liability risk),
  2. Investment risk (or asset risk),
  3. Credit risk (or risk of default),
  4. Operational risk
- Generally insurance companies have the know-how to manage and model their liability risk and are able to model the next two categories as well using standard finance models

Dependence Between Risks is Key

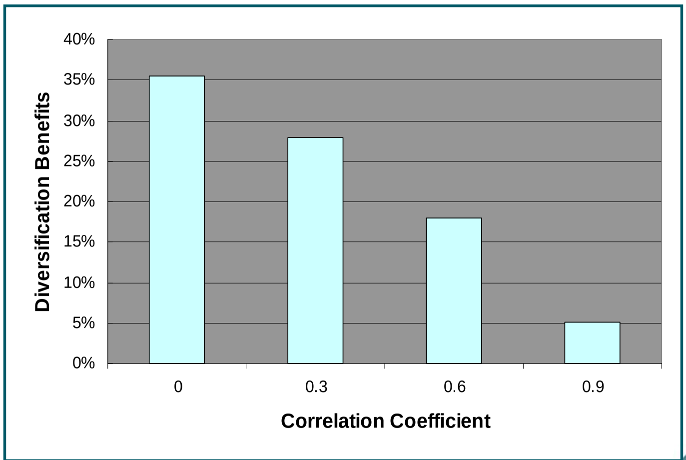
- *Risk Diversification* reduces a company's need for Risk-Adjusted Capital. This is key to both insurance and investments.
- However, risks are rarely completely independent:
  - Stock market crashes are usually not limited to one stock market.
  - Certain lines of business are affected by economic cycles, like aviation, credit & surety or life insurances.
  - Motor insurance is also correlated to motor liability insurance and both will vary during economic cycles.
  - Big catastrophes can produce claims in various lines of business.
- Dependence between risks *reduces the benefits* of diversification.
- The influence of dependence on the aggregated RAC is thus crucial and needs to be carefully analyzed.

Influence of Correlation on RAC

- Let us take the same risk twice (lognormally distributed, μ=10 and σ=1) and bundle them in a portfolio.
- Let us vary the correlation between the risks from 0 to 0.90.
- Here are the various diversification benefits, D, in percent:

$$D = 1 - \frac{RAC_P}{\sum_i RAC_i}$$

where RAC_P is the portfolio RAC and RAC_i are the RAC's of the various risks.



Dependence is not Always Linear

- We have learned to model dependence through *linear correlation*. The whole modern portfolio theory is based on correlation.
- Often dependence increases when diversification is most needed: in case of stress. It is thus *non-linear*.
- It is possible to use the *copulas* instead of linear correlation to model dependences (copula="generalized dependence structure" as opposed to "linear dependence"=correlation).
- The dependence structure will influence greatly the needs for RAC and the diversification benefits one can obtain.
- In the following, we present a statistical study of various dependence structures and their influence on diversification.

Agenda

Aim & Method

Aim

- To show the difficulty of statistically estimating the right dependence
- To illustrate the importance of using the correct Copula dependence when modeling dependent marginal distributions
- To analyze the influence of the dependence structure on the diversification benefits

Method

- Stochastic simulations and fitting of various dependence models
- To reproduce the behavior of the hierarchical dependence tree that is correlated through 3 Clayton copulas

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Basics of the Model (1/2)

- We use lognormal distributions as the basic risk of our portfolio:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x-\mu)^2 / 2\sigma^2} \qquad x \geq 0; \sigma > 0$$

- We choose μ=10 and σ=1 for all the risks*
- We want a simple risk model to study the influence of the dependence structure and function
- The basic risk is then used in various dependence configurations and with different dependence functions
- We choose a configuration that we assume to be the real one, which we fit with various other models

* Close to the parameters proposed by M. Bagarry For modeling insurance risks.

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Basics of the Model (2/2)

- RAC is calculated with Expected Shortfall (tVaR) for various risk tolerance levels. We summarize the results for the 1/100 tVaR.
- This is the risk measure used in the Swiss Solvency Test and at the basis of our own capital allocation model
- We also compute the VaR at 1/200 as it is the risk measure recommended by Solvency II and we compare both results

The Lognormal Distribution

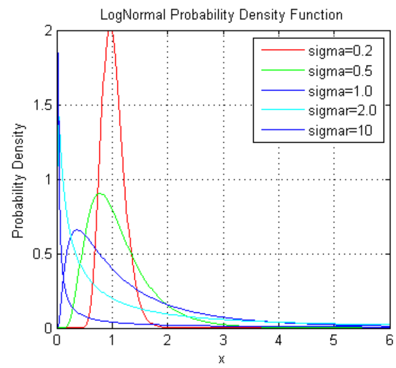
Lognormal distribution is the single-tailed probability distribution of any random variable whose logarithm is normally distributed.

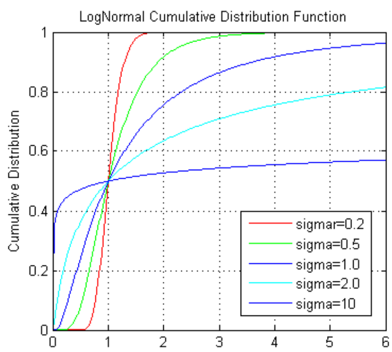
Lognormal Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x-\mu)^2/2\sigma^2} \quad x \geq 0; \sigma > 0$$

Lognormal Cumulative Distribution Function

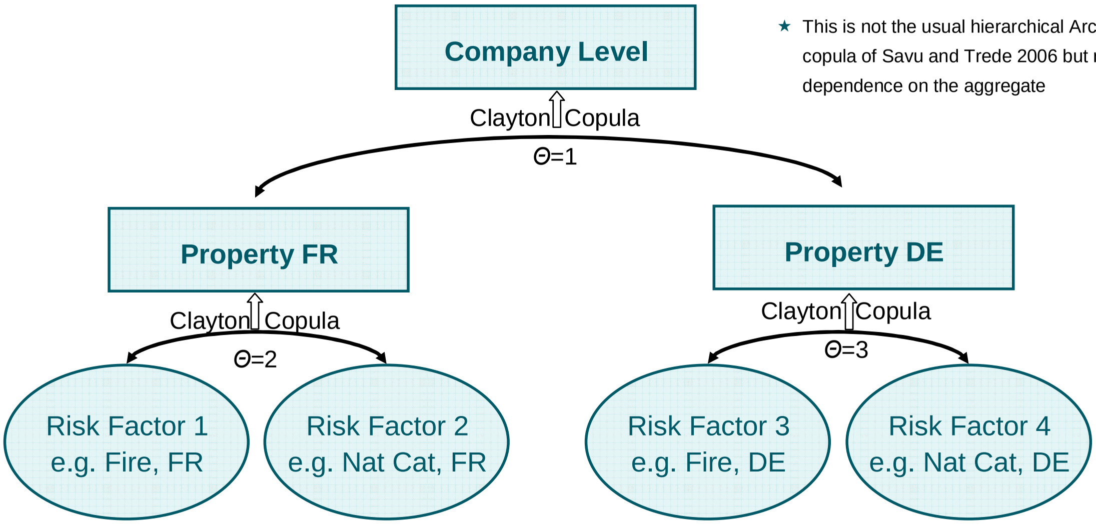
$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \quad x \geq 0; \sigma > 0$$





Hierarchical Dependence Tree

- **Base scenario**: Hierarchical Dependence Tree*
  - Hierarchy of 4 related marginal distributions
  - Using significantly different dependence parameters Θ



★ This is not the usual hierarchical Archimedean copula of Savu and Trede 2006 but rather a dependence on the aggregate

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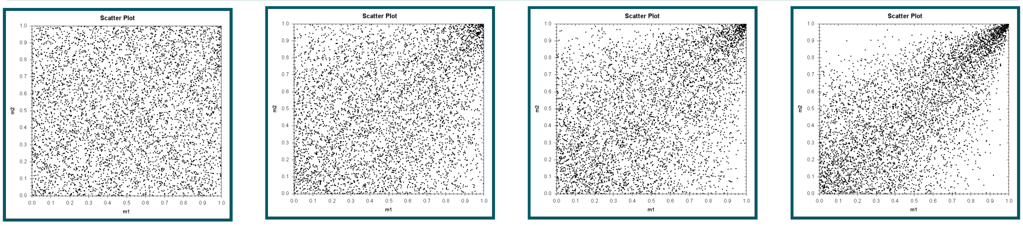
Archimedean Copula: Clayton Copula



θ = 0.1 θ = 0.5 θ = 1.0 θ = 2.0

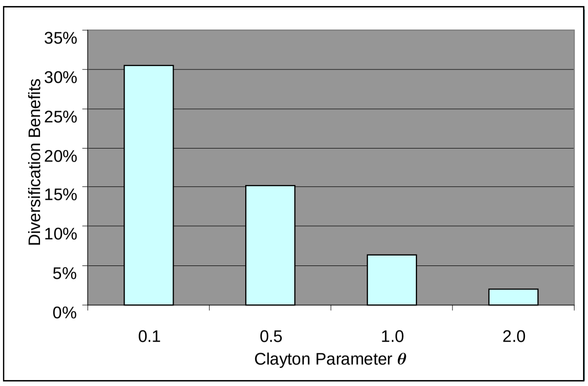
The Clayton Copula CDF is defined by:

$$C(u,v) = \max\left(\left[u^{-\theta} + v^{-\theta} - 1\right]^{-1/\theta}, 0\right)$$

With a *Generator* of the Copula:

$$\phi_\theta(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right)$$

The Clayton copula is Archimedean



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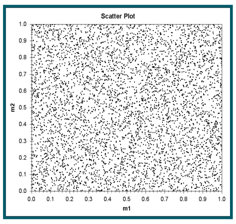
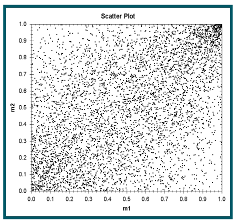
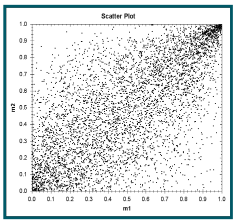
Archimedean Copula: Gumbel Copula



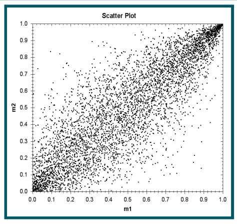
θ = 1.0



θ = 1.5



θ = 2.0



θ = 3.0

The Gumbel Copula CDF is defined by:

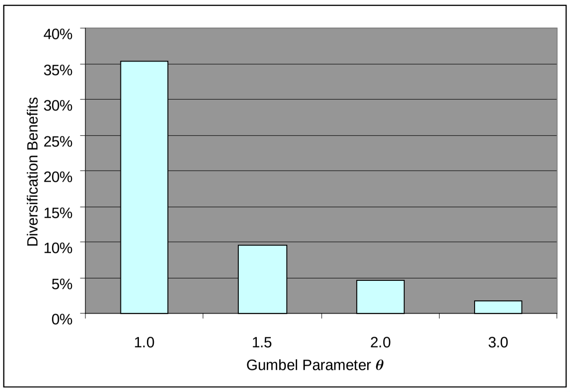
$$C(u,v) = \exp\left(-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{1/\theta}\right)$$

$$\theta \in [1, \infty)$$

where the *Generator* of the Copula is given by:

$$\phi_{\theta}(t) = (-\ln t)^{\theta}$$

The Gumbel copula is Archimedean



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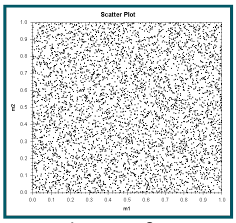
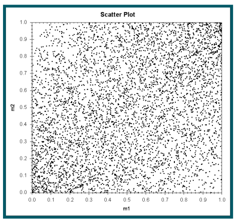
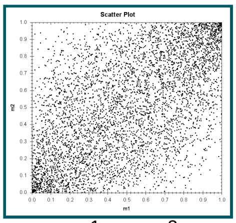
Elliptical Copula: Rank Correlation



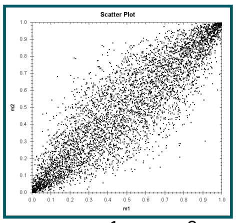
	m1	m2
m1	1	0
m2	0	1



	m1	m2
m1	1	0.3
m2	0.3	1



	m1	m2
m1	1	0.6
m2	0.6	1



	m1	m2
m1	1	0.9
m2	0.9	1

The multivariate Normal distribution copula has a matrix as a parameter. The PDF of a Normal copula is:

$$c_R^{Normale}(u_1, u_2, \dots, u_n) = \frac{1}{|R|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2} \cdot S' \cdot (R^{-1} - I) \cdot S\right)$$

where, $S_j = \Phi^{-1}(u_j)$, Φ^{-1} is the inverse of the CDF N(0,1) and *I* is the identity matrix of size *n*.

The rank correlation is an elliptical copula.

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Elliptical Copula: Student's T

The multivariate Student's T distribution copula also has a matrix as a parameter.
The PDF of a Student's T copula is:

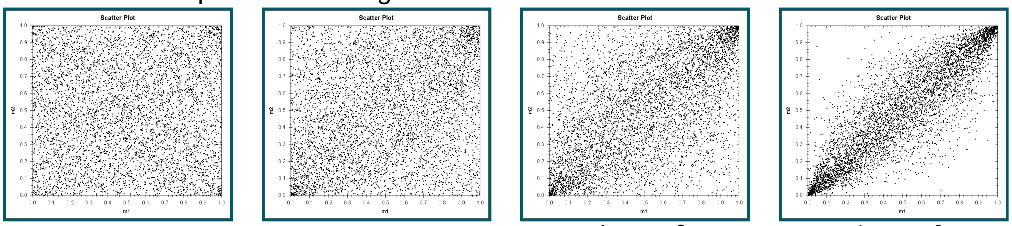
$$c^{\text{Student}}_{R,\nu}(u_1,u_2,\cdots,u_n)=|R|^{-\frac{1}{2}}\cdot\frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\cdot\left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right)^{n}\cdot\frac{\left(1+\frac{1}{\nu}\cdot S'\cdot R^{-1}\cdot S\right)^{-\frac{\nu+n}{2}}}{\prod_{j=1}^{n}\left(1+\frac{S_j^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

Where $S_j = T_\nu^{-1}(u_j)$, T_ν^{-1} is the inverse of the CDF of the univariate T distribution with ν degrees of freedom

The Student's T copula is an elliptical copula

Effects of matrix parameter changes:

ν =3



	m1	m2
m1	1	0
m2	0	1

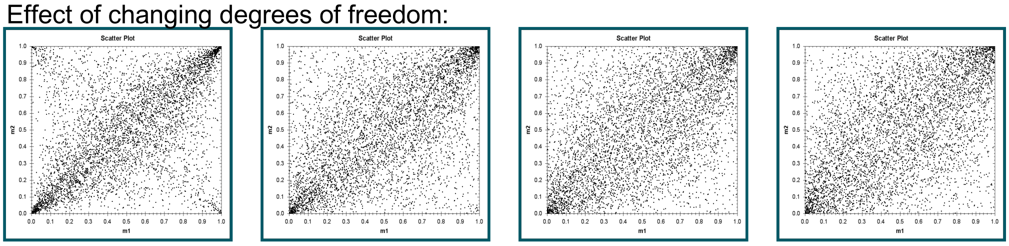
	m1	m2
m1	1	0.3
m2	0.3	1

	m1	m2
m1	1	0.6
m2	0.6	1

	m1	m2
m1	1	0.9
m2	0.9	1

Student's T

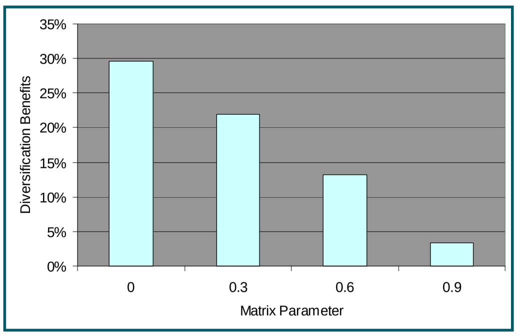
Effect of changing degrees of freedom:



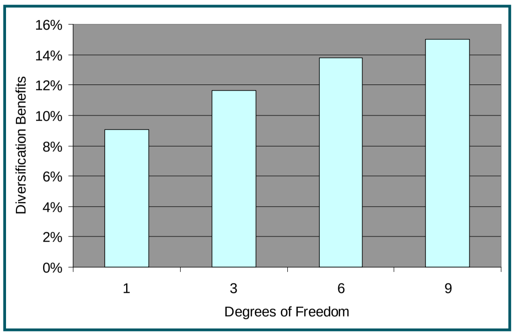
ν = 1 ν = 3 ν = 6 ν = 9

	m1	m2
m1	1	0.6
m2	0.6	1

Diversification benefits from varying the matrix parameter (3 degrees of freedom):



Diversification benefits from varying the degrees of freedom:

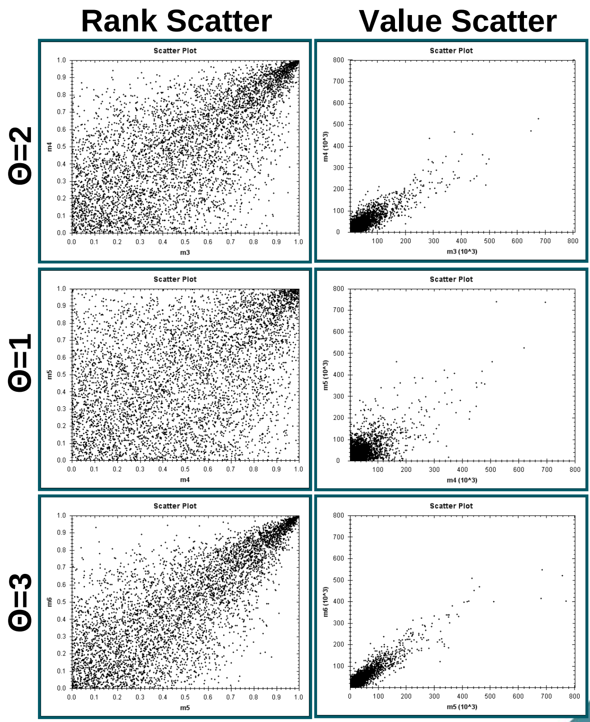


Value versus Rank Scatter

Significance of Value versus Rank Scatter

- Value scatter is used to characterize the spread of the marginal distributions
- Rank scatter shows the underlying dependence between the marginals

For the purpose of analysing dependence we shall display only rank scatter from here on



Agenda

1	Risk-Adjusted Capital and dependence
2	Description of the model
3	Influence of the number of observations on calibration
4	Influence of dependence structure and models
5	Ways to model the dependence
6	Conclusion

Fitting Scenarios

Scenarios: We fit the original dependence using the following copula scenarios and calculate the Diversification and RAC

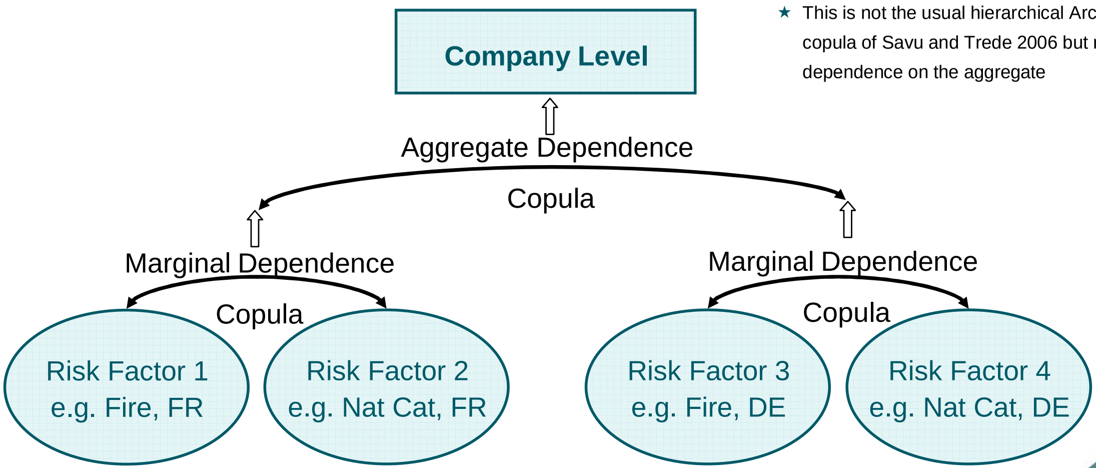
- Hierarchical Dependence
  - Gumbel Hierarchy
  - Rank Correlation Hierarchy
  - Student-T Hierarchy
- Flat Dependence
  - Clayton Flat
  - Gumbel Flat
  - Rank Correlation Flat
  - Student-T Flat
- The base scenario is the *hierarchical Clayton* scenario presented before

Fitting: Hierarchical Scenario

Fitting Scenario

- The Clayton Hierarchical Tree is fit by using the same structure* correlated with different copulas for each scenario
- The results are then displayed on a rank-scatter plot and through diversification and RAC value calculation

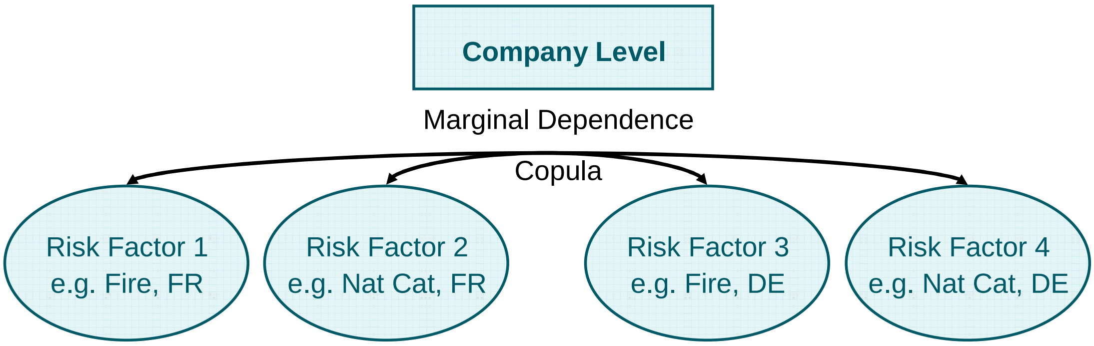
* This is not the usual hierarchical Archimedean copula of Savu and Trede 2006 but rather a dependence on the aggregate



Fitting: Flat Scenario

Business Scenario:

- Small company, with a small amount of business in each basket ➔ baskets are merged.
- All marginals modeled by *one copula*



Estimation of Copulas

The following estimation methods were used:

Clayton / Gumbel:
Maximum likelihood estimation, i.e. estimates the parameter θ by maximizing the log-likelihood function $\mathrm{LMLF} = \sum_i \log(c(\mathbf{x_i}))$, where $c(\mathbf{x_i})$ is the copula density of point *i*

Rank Correlation:
Estimate the Spearman's correlation $\mathrm{RankCorr}(X,Y) = 12\mathrm{E}[(F_X(X)-0.5)(F_Y(Y)-0.5)]$ for each pair X and Y. The correlation matrix for the Gauss copula can be derived as

$$\rho_{ij} = 2\sin\left(\frac{\pi}{6}\mathrm{RankCorr}(X_i, X_j)\right)$$

Student's T:
Estimate the Kendall $\tau(X,Y) = \frac{2}{N(N-1)}\sum_{i<j}\mathrm{sign}[(X_i - X_j)(Y_i - Y_j)]$

The correlation matrix can be derived as $\rho_{ij} = \sin\left(\frac{\pi}{2}\tau_{ij}\right)$

The degree of freedom ν is estimated with *maximum likelihood*.

Fitting Convergence Plots: Methodology

The Fitting Convergence Plots are drawn using the following methodology:

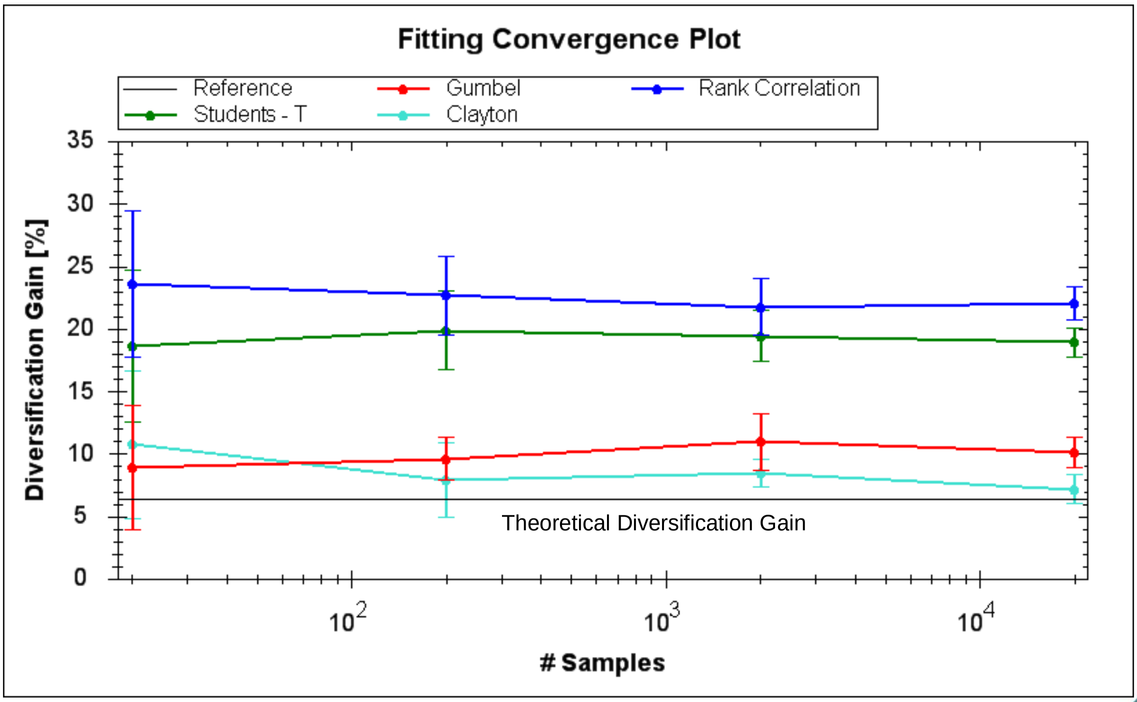
1. *Simulate* *N* observations from the reference scenario
2. *Fit* the corresponding scenario to the *N* observations
3. *Resample* the fitted scenario with 50'000 observations
4. *Measure* the diversification gain

Repeat 10 times

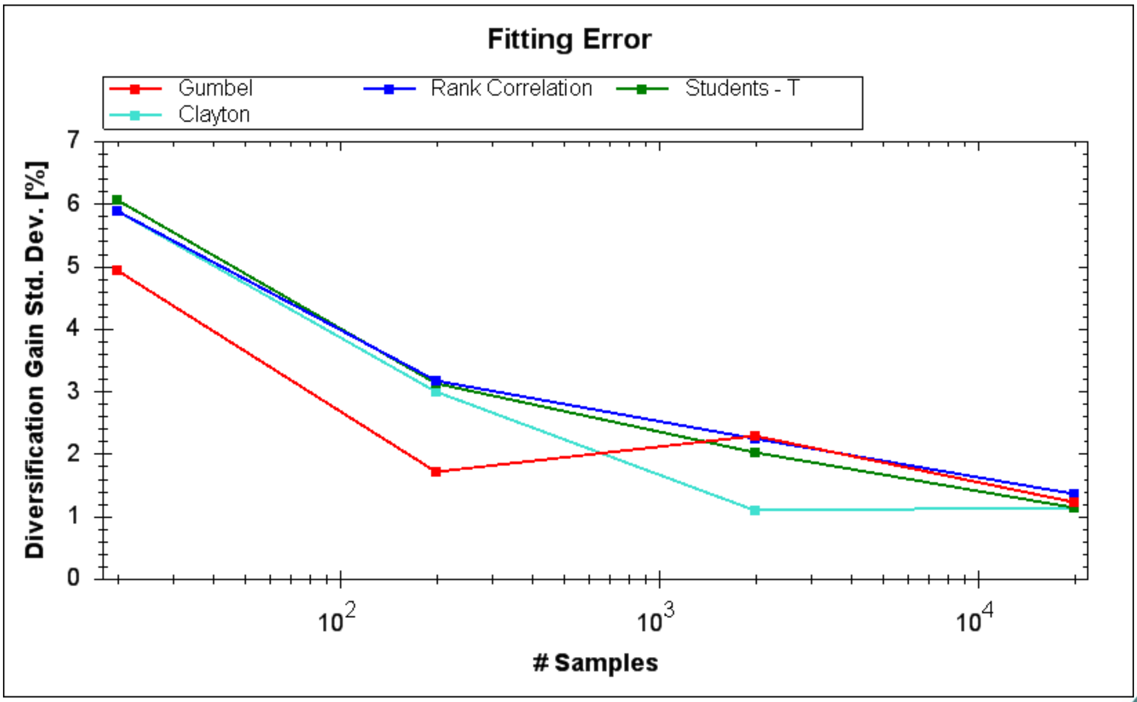
The mean and standard deviation of the 10 runs per point are calculated.

- The fitting convergence plots show the mean ± one standard deviation for each *N*.
- The fitting error plots show the standard deviation for each *N*.

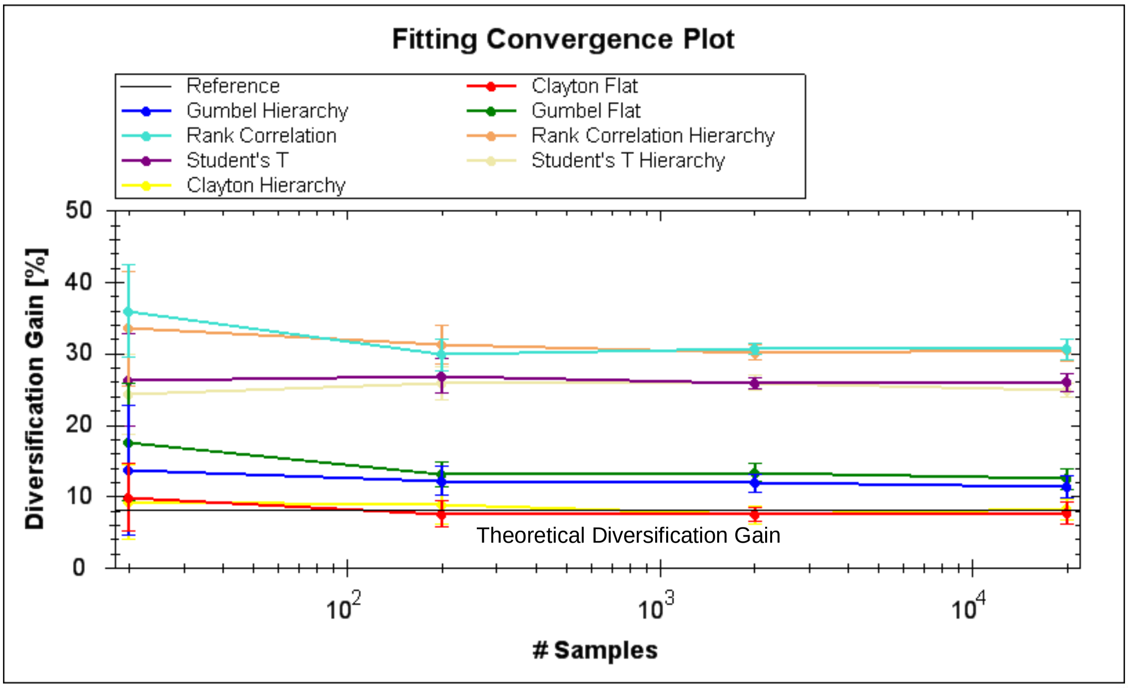
Convergence of Fits for 2 Marginals Starting from Clayton θ = 1



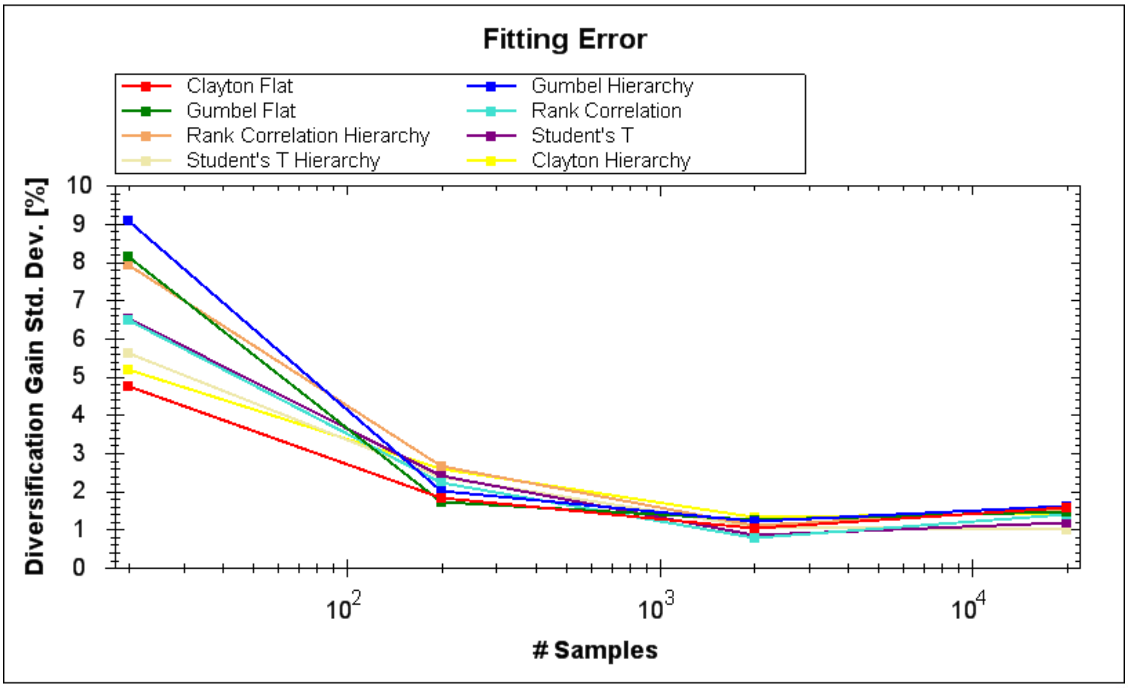
Standard Deviations of the Fits for 2 Marginals Starting from Clayton θ = 1



Convergence of the Fits for 4 Marginals



Standard Deviation of the Fits for 4 Marginals



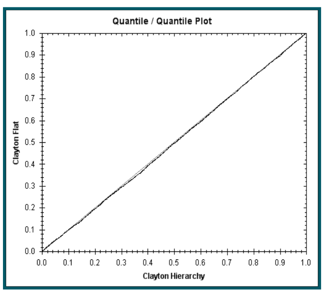
Number of Observations matters less than Dependence Models

- We see that the elliptical copulas keep a systematic bias whatever the number of observations
- The Archimedean copulas fit much better the theoretical value with Clayton doing it the best, as expected
- The error of the estimation decreases with the number of observations and remains at a certain level even with 50’000 observations
- The structure of the dependence (hierarchical or flat) does not affect really the diversification benefit with hierarchical being slightly better for Archimedean copulas
- When the dependence is asymmetric (as it is usually the case for insurance liabilities), it is difficult to model it with symmetric dependence models (use asymmetric copulas)

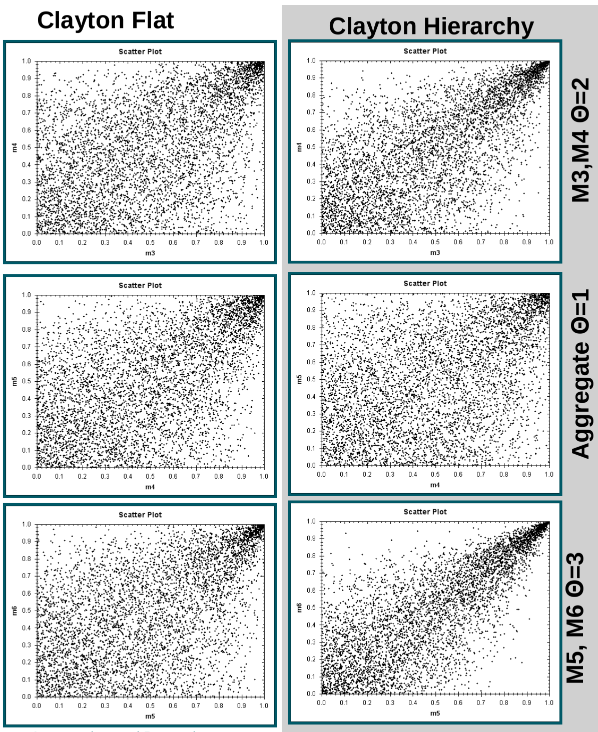
Agenda

Results of Fit: Clayton Flat

- As expected, the two stronger dependences are reduced and the weaker dependence is strengthened.
- Fit: $\theta = 1.2$
- The RAC and diversification gain are even slightly more conservative. The error is almost negligible.
- The Q/Q plot shows good agreement of the models.

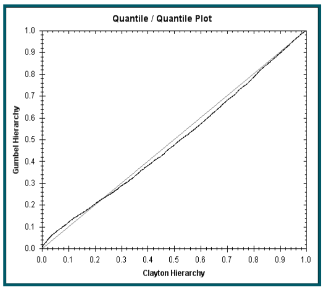


	Clayton Hierarchy	Clayton Flat
Expected	145'550	145'407
Std Dev	172'587	172'948
LH Div. Gain	9.56%	9.14%
RH VaR	1'070'462	1'078'190
RH Shortfall	1'248'146	1'251'138
RH RAC	1'102'595	1'105'731
RH Div. Gain	8.17%	7.73%

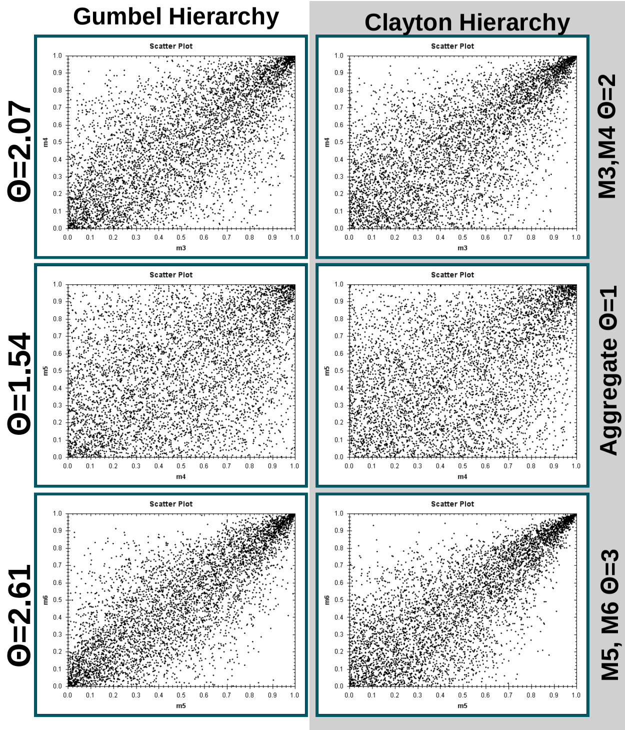


Results of Fit: Gumbel Hierarchy

- As the Clayton, the Gumbel copula is an Archimedean Copula.
- Gumbel represents the Clayton well in the tail region for all 3 distributions
- In contrast to the Clayton, the Gumbel introduces a dependence also in the lower tail.
- RAC is slightly underestimated
- The Q/Q shows fair similarity of the two Copula types.

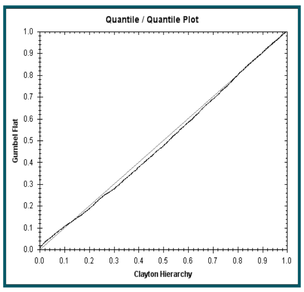


	Clayton Hierarchy	Gumbel Hierarchy
Expected	145'550	144'964
Std Dev	172'587	164'507
LH Div. Gain	9.56%	5.70%
RH VaR	1'070'462	1'021'144
RH Shortfall	1'248'146	1'195'423
RH RAC	1'102'595	1'050'458
RH Div. Gain	8.17%	11.73%

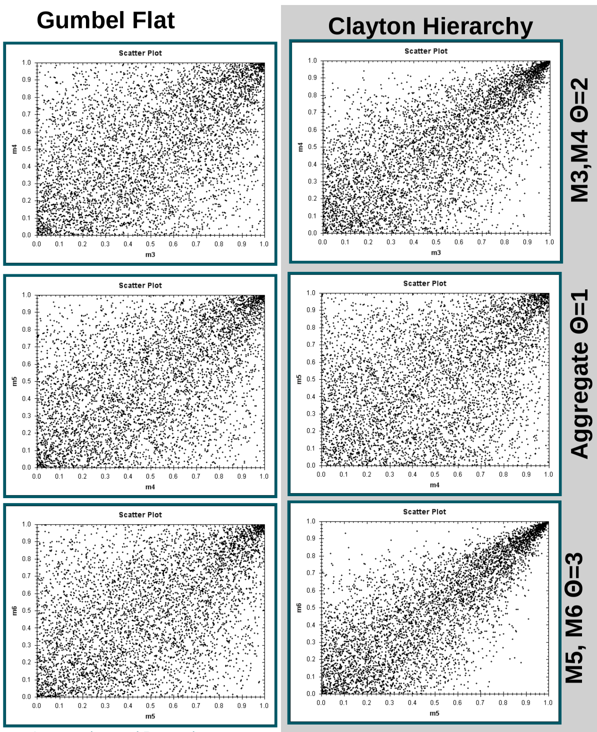


Results of Fit: Gumbel Flat

- The coupling on both ends is again visible
- Overall, the fit of the tail region is still reasonable
- Since the dependence of the lower tail reduces the tail dependence in the upper tail in the fit, the RAC is slightly underestimated.
- The Q/Q plot still shows a reasonably good agreement between the copulas.

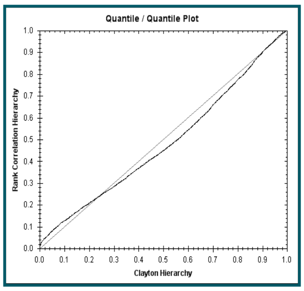


	Clayton Hierarchy	Gumbel Flat
Expected	145'550	145'063
Std Dev	172'587	162'936
LH Div. Gain	9.56%	7.20%
RH VaR	1'070'462	1'028'533
RH Shortfall	1'248'146	1'188'803
RH RAC	1'102'595	1'043'740
RH Div. Gain	8.17%	12.69%



Results of Fit: Rank Correlation Hierarchy

- Rank correlation is symmetric ➔ strong correlation also for the lower tail.
- The upper tail is much less pointed than for the Clayton.
- The RAC is substantially underestimated.
- The diversification gain is unrealistically high.
- The Q/Q plot shows the deviation in the tails.



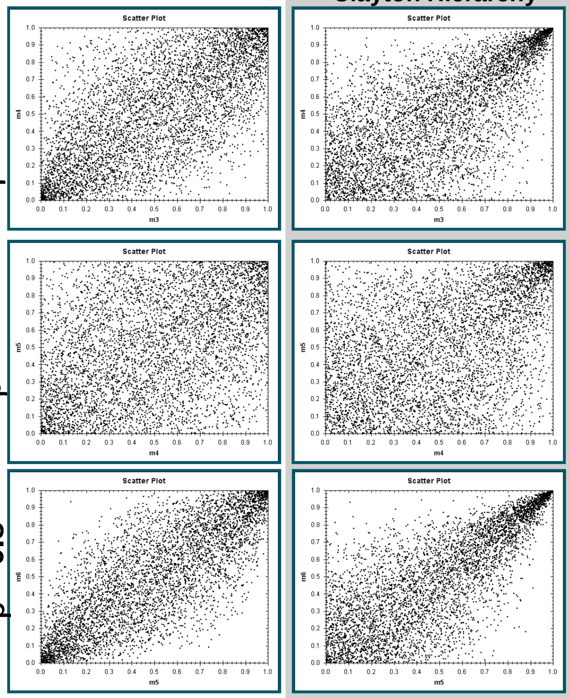
	Clayton Hierarchy	RankCorr. Hierarchy
Expected	145'550	145'464
Std Dev	172'587	146'133
LH Div. Gain	9.56%	3.90%
RH VaR	1'070'462	877'052
RH Shortfall	1'248'146	990'103
RH RAC	1'102'595	844'639
RH Div. Gain	**8.17%**	**30.33%**

Rank Corr. Hierarchy | Clayton Hierarchy

$\rho = 0.7$ | M3,M4 Θ=2

$\rho = 0.5$ | Aggregate Θ=1

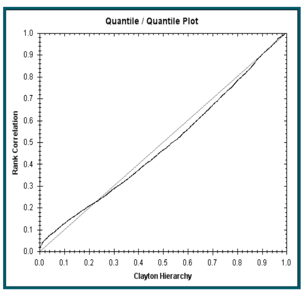
$\rho = 0.8$ | M5, M6 Θ=3



Results of Fit: Rank Correlation Flat

- The flat rank correlation produces alsost the same results as the hierarchical one.
- This can be seen in all graphics as well as the RAC calculations.

$$\rho_{i,j} = \begin{pmatrix} 1.00 & 0.70 & 0.45 & 0.46 \\ 0.70 & 1.00 & 0.45 & 0.46 \\ 0.45 & 0.45 & 1.00 & 0.80 \\ 0.46 & 0.46 & 0.80 & 1.00 \end{pmatrix}$$



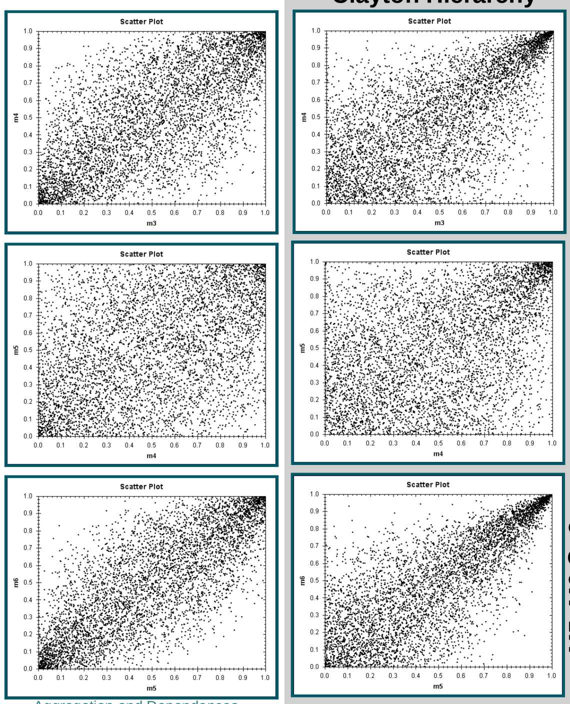
	Clayton Hierarchy	Rank Corr.
Expected	145'550	145'293
Std Dev	172'587	144'448
LH Div. Gain	9.56%	3.98%
RH VaR	1'070'462	870'389
RH Shortfall	1'248'146	977'617
RH RAC	1'102'595	832'324
RH Div. Gain	**8.17%**	**30.70%**

Rank Corr. Flat | Clayton Hierarchy

M3,M4 Θ=2

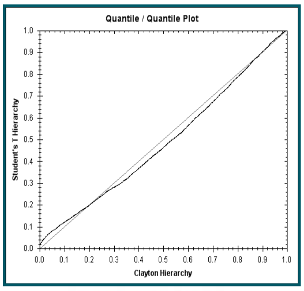
Aggregate Θ=1

M5, M6 Θ=3



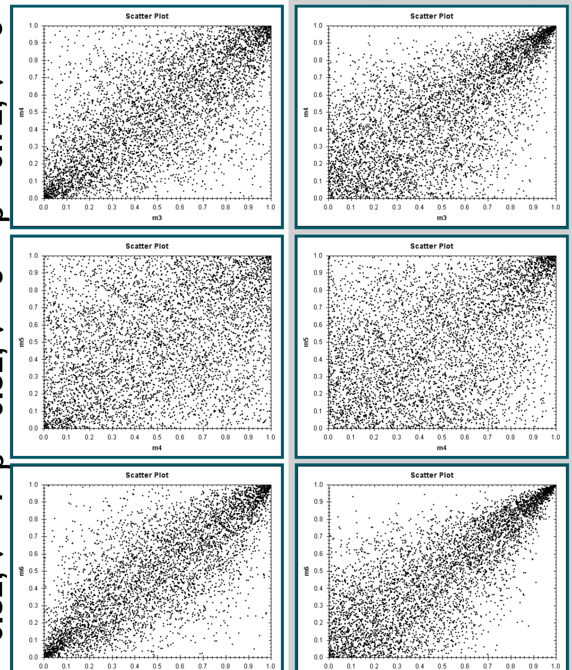
Results of Fit: Student's T Hierarchy

- As the Rank Correlation, the Student's T copula is an elliptical copula.
- The Student's T has one parameter more per dependence than the Rank Correlation
- Dependence is symmetric, i.e. also introduced in the lower tail.
- RAC is substantially underestimated.
- Unrealistically high diversification gain
- The Q/Q plot looks similar as for the Rank Correlation.



	Clayton Hierarchy	Student T Hierarchy
Expected	145'550	145'088
Std Dev	172'587	150'290
LH Div. Gain	9.56%	3.05%
RH VaR	1'070'462	909'335
RH Shortfall	1'248'146	1'045'099
RH RAC	1'102'595	900'012
RH Div. Gain	8.17%	25%

Student-T Hierarchy | **Clayton Hierarchy**



$\rho = 0.71;\ \nu = 6$ — **M3,M4 Θ=2**

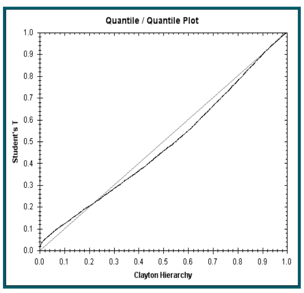
$\rho = 0.51;\ \nu = 9$ — **Aggregate Θ=1**

$\rho = 0.81;\ \nu = 4$ — **M5, M6 Θ=3**

Results of Fit: Student-T Flat

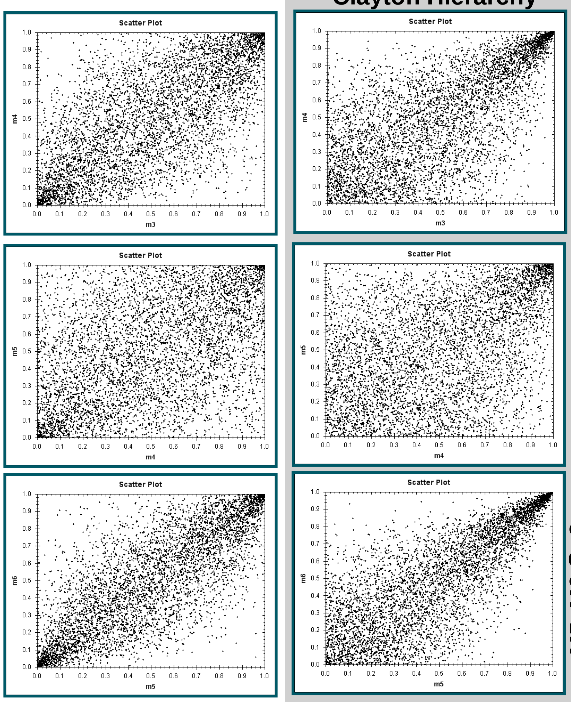
- Similar to Student-T Hierarchy
- Has one parameter more than the Rank Correlation ➔ Slightly better
- Has two parameters less than the hierarchical Student's T ➔ Slightly worse

$$\rho_{i,j} = \begin{pmatrix} 1.00 & 0.71 & 0.45 & 0.46 \\ 0.71 & 1.00 & 0.45 & 0.46 \\ 0.45 & 0.45 & 1.00 & 0.81 \\ 0.46 & 0.46 & 0.81 & 1.00 \end{pmatrix} \qquad \nu = 10$$



	Clayton Hierarchy	Student's T
Expected	145'550	144'980
Std Dev	172'587	150'838
LH Div. Gain	9.56%	3.20%
RH VaR	1'070'462	901'757
RH Shortfall	1'248'146	1'043'301
RH RAC	1'102'595	898'322
RH Div. Gain	8.17%	25.42%

Student-T Flat | **Clayton Hierarchy**



M3,M4 Θ=2

Aggregate Θ=1

M5, M6 Θ=3

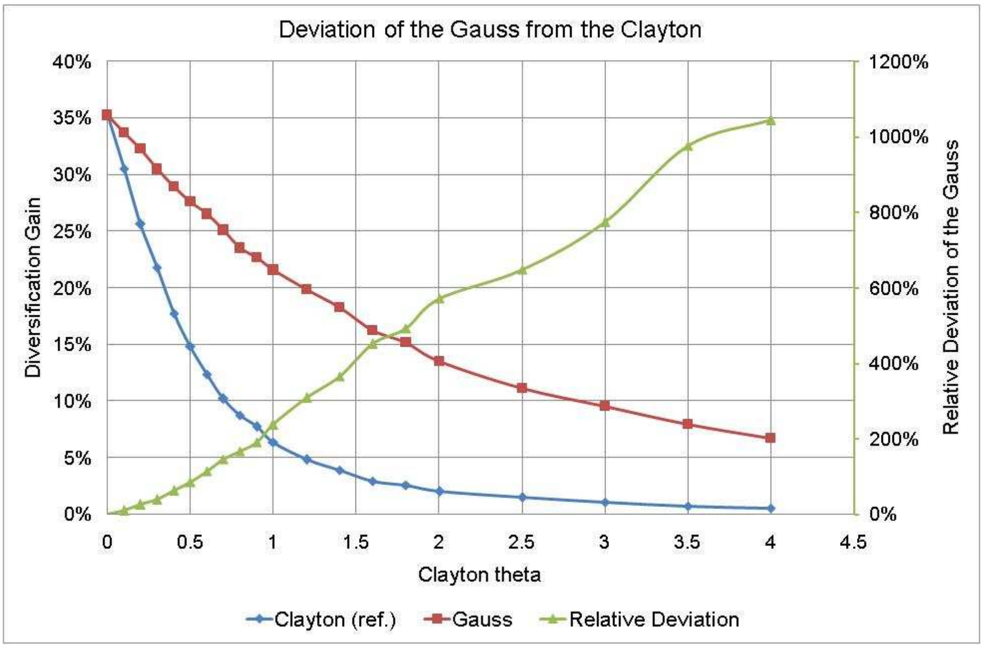
Summary of the Statistical Results

Statistical Results	250,000 simulations							
	Clayton Hierarchy	Clayton Flat	Gumbel Hierarchy	Gumbel Flat	Rank Corr.	Rank Corr. Hierarchy	Student's T	Student's T Hierarchy
Expected	145‘550	145‘407	144‘965	145‘063	145‘293	145‘464	144‘980	145‘088
Std Dev	172‘587	172‘948	164‘507	162‘936	144‘448	146‘133	150‘838	150‘290
LH Div. Gain	9.56%	9.14%	5.70%	7.20%	3.98%	3.90%	3.20%	3.05%
RH VaR	1‘070‘462	1‘078‘190	1‘021‘144	1‘028‘533	870‘389	877‘052	901‘757	909‘335
RH Shortfall	1‘248‘146	1‘251‘138	1‘195‘423	1‘188‘803	977‘617	990‘103	1‘043‘301	1‘045‘099
RH RAC	1‘102‘595	1‘105‘731	1‘050‘458	1‘043‘740	832‘324	844‘639	898‘322	900‘012
RH Div. Gain	**8.17%**	**7.73%**	**11.73%**	**12.69%**	**30.70%**	**30.33%**	**25.42%**	**25%**

Dependence Model matters more than Dependence Structure

- We see that the elliptical copulas do not improve by moving from a flat structure to a hierarchical one.
- The Gumbel copula slightly improves if used in the appropriate dependence structure.
- The elliptical copulas grossly underestimate the risk and show undue diversification benefits.
- Gumbel copula is able to produce reasonable results on the left tail but also emphasizes a dependence on the right tail that does not exist in the benchmark model.

Underestimation starts already with weak dependence



Agenda

How to Estimate Dependences?

Dependences can hardly be described by one number such as a linear correlation coefficient.

We just saw that it is possible to use the ***copulas*** to model dependences.

In insurance, there is often not enough liability data to estimate the copulas.

Nevertheless, copulas can be used to translate an opinion about dependences in the portfolio into a model:

- Select a copula with an appropriate shape
  - *increased dependences in the tail*
    - this feature is observable in historic insurance loss data
- Try to estimate conditional probabilities by asking questions such as "What if a particular risk turned very bad?"
  - Think about *adverse scenarios* in the portfolio
  - Look at *causal relations* between risks

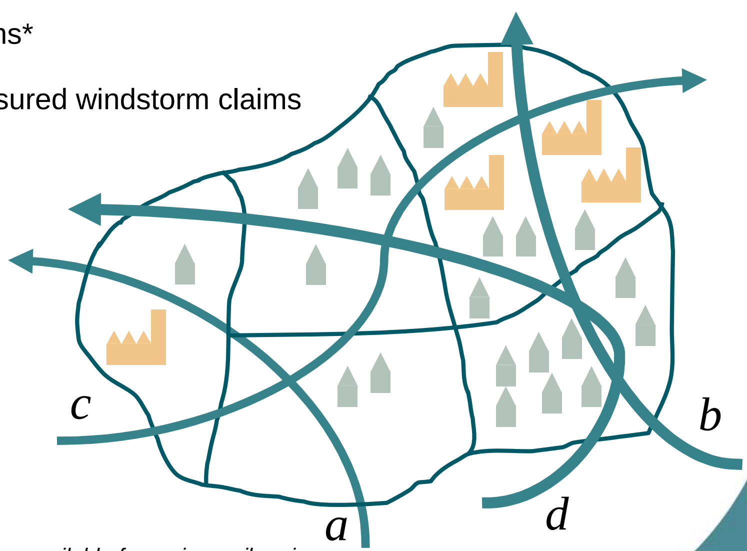
Example: Windstorm

Collect the exposures from all policies per zip-code area in an accumulation control system

- Here: Private homes and industrial plants
- Scenarios = Windstorms*
- Random Variable = insured windstorm claims

Stochastic Simulation

Scenario	Insured Loss
a	3
b	27
c	11
d	8



**There are commercial models of this type available for major peril regions.*

Describing Dependences

Scenario based simulation

- Dependences between random variables modeled on the same scenarios is incorporated automatically
- Example: dependence in "our" windstorm model between losses on industrial risks and on private home owners
- Building a realistic model of that type is challenging

Distribution based simulation

- Via *joint* simulations of the individual distribution
- Dependent sampling of the joint uniform random numbers
  - ➔ *copula*
- Calibration is an issue

Strategy for modeling dependences

- Using the knowledge of the underlying business, develop a hierarchical model for dependences in order to reduce the parameter space and describe more accurately the main sources of dependent behavior
- Wherever we know a causal dependence, we model it explicitly
- Systematically usage of non-symmetric copulas: Clayton copula
- Wherever there is enough data, we calibrate statistically the parameters
- In absence of data, we use stress scenarios to estimate conditional probabilities

Agenda

Summary

- Neglecting dependences leads to a *gross underestimation* of the Risk-Adjusted Capital
- Neglecting the non-linearity of dependences leads also to an *overestimation* of the diversification benefits
- Dependences in insurance risks are usually *asymmetric*: stronger on the negative side than on the positive one
- A suitable copula to model those type of dependences is the Clayton copula
- To get the right diversification benefit the choice of the right dependence model matters most
- With a relatively modest number of data it is possible to obtain a reasonable estimate of the diversification benefit with Clayton copula

Further research

- Develop the empirical exploration of dependences
- Analyze retarded dependences: causal relations
- Pursue the study of the influence of the hierarchical tree on total RAC with more branches and depth