

## Theory of self-organized criticality for problems with extremal dynamics

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**Abstract.** – We introduce a general theoretical scheme for a class of phenomena characterized by an extremal dynamics and quenched disorder. The approach is based on a transformation of the quenched dynamics into a stochastic one with cognitive memory and on other concepts which permit a mathematical characterization of the self-organized nature of the avalanche type dynamics. In addition it is possible to compute the relevant critical exponents directly from the microscopic model. A specific application to invasion percolation is presented but the approach can be easily extended to various other problems.

Extremal models of self-organized criticality (SOC [1]) have recently attracted significant theoretical attention. This class of models describes phenomena ranging from fluid displacement in porous disordered media [2], to punctuated biological evolution [1]. In these models, at each time step the dynamical activity is concentrated on the site with the *extremal* value of a quenched disordered variable on a certain growth interface. This rule leads to a rich and complex behaviour, which opens theoretical problems of new type [3].

In this letter we describe a general theoretical approach which addresses the essential properties of extremal models: i) the understanding of the scale-invariance and self-organization; ii) the origin of the avalanche dynamics and iii) the computation of the relevant critical exponents. We apply it specifically to invasion percolation (IP), but it can be easily extended to other models of this type like the Bak and Sneppen model [4].

The theoretical methods for fractal growth, like Fixed Scale Transformation [5], cannot address the problem of the irreversible dynamics with *quenched* disorder because they are based on a stochastic dynamics with a local growth probability well defined at each time step. In order to overcome this basic problem, we introduced a mapping of a quenched extremal dynamics into a stochastic one with cognitive memory, the *quenched-stochastic transformation*, also called Run Time Statistics (RTS) [6]. This approach was developed in various stages [7], [6], [4] and we can now formulate it as a general theoretical scheme. Its essential points are

- The quenched-stochastic transformation.
- The identification of the microscopic fixed point dynamics (SOC).

- The identification of the scale invariant dynamics for block variables.
- The definition of *local* growth rules for the extremal model. This clarifies the origin of *avalanche dynamics*.
- The use of the above elements in the FST scheme to compute analytically the relevant exponents of the model.

Let us start with the quenched-stochastic transformation. We will discuss it for the case of bond invasion percolation (IP) [2]. The IP model describes the capillary displacement of a fluid in a porous medium. The medium is represented as a network of bonds. To each bond  $i$  is assigned a quenched random number  $x_i$  extracted from a given distribution with density  $\rho(x) = 1$ . The invading cluster evolves by occupying the bond with the smallest  $x_i$  on its perimeter. The basic idea of the RTS is to map the deterministic extremal IP dynamics into an annealed stochastic process. A general stochastic process is based on the following elements: a) a Growth Probability Distribution (GPD)  $\{\mu_{i,t}\}$  for the growth of bond  $i$  at time  $t$ ; b) a rule for the evolution of the probability distribution during the growth process.

The original IP process is instead deterministic for each configuration of the disorder and the average over this should be taken at the end of the process. We now illustrate how to identify an effective stochastic process which corresponds to the IP dynamics.

An insight into the essence of the question is given by the following example. Consider two independent random variables  $X_1, X_2$  uniformly distributed in  $[0, 1]$  and let us eliminate the smallest, for example  $X_2$ . The probability that  $X_2 < X_1$  is  $1/2$ . At the second “time step”, we compare the surviving variable  $X_1$  with a third, uniform, random variable  $X_3$  just added to the game and, again, we eliminate the smallest one. In this case we need to compute the probability  $\mu_3$  that  $X_3 < X_1$  *given* that  $X_2 < X_1$ . This, using the rules of conditional probability, reads

$$\begin{aligned} \mu_3 &= \tilde{P}(X_3 < X_1) = P(X_3 < X_1 | X_2 < X_1) = \\ &= \frac{P(X_3 < X_1 \cap X_2 < X_1)}{P(X_2 < X_1)} = \frac{2}{3}, \end{aligned} \quad (1)$$

where  $P(A|B)$  is the probability of the event  $A$ , given that  $B$  occurred, and  $P(A \cap B)$  is the probability of occurrence of both  $A$  and  $B$ . Equation (1) tells us that the information  $X_2 < X_1$  *changes in a conditional way* the *effective* probability density  $p_1(x)$  of  $X_1$ . In fact, by imposing the condition  $X_2 < x$  (given  $x < X_1 < x + dx$ ) we get  $p_1(x) = 2x$ . Qualitatively, the event  $X_2 < X_1$  decreases the probability that  $X_1$  has small values.

This is the essential idea of the *quenched-stochastic transformation*. Let us come back to the IP model. In view of the above example, each perimeter variable will have, at time  $t$ , an effective density depending on the past growth history of the bond. If a variable has lost many times, it will have a density more and more concentrated on great values. The past history of a variable can be represented by only one parameter, the number of time steps  $k$  that the variable has spent in the perimeter, that we call “age” of the variable. At time  $t$ , variables with the same age  $k$  will have the same effective density  $p_{k,t}(x)$  ( $p_{0,t}(x) = \rho(x)$ ). We can now express the probability that a variable of age  $k$  is selected at time  $t$ , *i.e.* it is the smallest perimeter variable, given the effective densities  $p_{k,t}(x)$  of all the perimeter variables for a realization of the process at time  $t$ :

$$\mu_{k,t} = \int_0^1 dx p_{k,t}(x) \prod_{\theta} (1 - P_{\theta,t}(x))^{n_{\theta,t} - \delta_{\theta,k}}, \quad (2)$$

where  $P_{k,t}(x) = \int_0^x dy p_{k,t}(y)$ . The product in eq. (2) is intended over all the ages  $\theta$  of the variables and  $n_{\theta,t}$  is the number of active variables of *age*  $\theta$  at time  $t$ . The expression inside the integral takes into account the competition of the selected variable with each one of the other

active variables, while the integral between 0 and 1 takes into account all the possible values of the growing variable. A similar reasoning, that is in [8], leads to the temporal evolution of the densities of the still active (perimeter) variables:

$$p_{\theta+1,t+1}(x) = p_{\theta,t}(x) \int_0^x \frac{m_{k,t}(y)}{1 - P_{\theta,t}(y)} dy, \quad (3)$$

where  $m_{k,t}(y) = \left[ p_{k,t}(x) \prod_{\theta} (1 - P_{\theta,t}(x))^{n_{\theta,t} - \delta_{\theta,k}} \right] / \mu_{k,t}$ .

Equations (2), (3) describe a quenched extremal process as a stochastic process with memory. The presence of memory is enlightened by the dependence of the GPD on  $k$ . A mean-field like expansion [6] of eq. (2) in the limit  $t \rightarrow \infty$  gives:  $\mu_{k,\infty} \sim 1/(k+1)^\alpha$ . The power law behaviour of  $\mu_{k,t}$  guarantees that screening is preserved at all scales, which is the condition to generate fractal structures [9].

An important quantity to study in IP is the histogram  $\Phi_t(x)$ , which is the distribution at time  $t$  of the perimeter variables. It has been shown numerically [2] that the histogram of IP self-organizes asymptotically into a stable shape, a theta-function with a discontinuity at  $x = p_c < 1$ , where  $p_c$  is the critical bond percolation threshold. In ref. [6] an equation for the temporal evolution of  $\Phi_t(x)$  is derived analytically directly from the RTS microscopic dynamics, without *a priori* assumptions. The final form of this *histogram equation*, obtained after some technical approximations [6] is

$$\partial_x \Phi_t(x) = \beta \Omega_t \Phi_t^2(x) \left[ 1 - \frac{\omega_t}{\omega_t + 1} \Phi_t(x) \right], \quad (4)$$

where  $N_t$  is the number of perimeter variables a time  $t$  for a realization of the growth process,  $\Omega_t = \langle N_t \rangle$ ,  $\omega_t = \langle N_{t+1} - N_t \rangle$ , the mean  $\langle \dots \rangle$  is over different realizations, and  $\beta$  is the solution of:  $\beta = 1 - e^{-\beta \Omega_t}$ . The solution of eq. 4 converges asymptotically to a theta-function (fig. 1):

$$\lim_{t \rightarrow \infty} \Phi_t(x) = \frac{1}{1 - p_c} \theta(x - p_c), \quad (5)$$

where  $p_c = 1/2$  for  $2 - d$  bond IP. A similar result is obtained by simulations in [10] and by a rigorous reasoning in [11]. This result clarifies the SOC nature of the problem and it will be crucial in the definition of avalanche dynamics. In this direction, similar conclusions have been obtained by Bak *et al.* [3], on a phenomenological basis.

The fact that the asymptotic self-organized histogram (eq. (5)) converges to a theta-function implies that *almost all* the perimeter bonds have quenched variables with values between  $p_c$  and 1. This brings us to the mathematical conditions for the avalanche dynamics which can be expressed as follows: a bond that grows is the *initiator* of an avalanche if there is a finite probability that the next growth process corresponds to a younger bond connected to the previous one even in the limit of an infinite system. Clearly the theta-function in eq. (5) guarantees that this is indeed the case for IP. On the contrary in stochastic growth processes like DLA and DBM the growth is smooth and it does not proceed by avalanches.

The next step is the identification of the scale-invariant dynamics for block variables. In fig. 2 we show a coarse-graining procedure for the extremal dynamics of IP. In the left side of the figure we show some paths leading to cell  $B$  (or  $A$ ), each one composed by a set of quenched variables ( $\{\epsilon_i\}_b$ ) and characterized by the largest variable ( $\epsilon_b$ ) in this set. The best path leading to cell  $B$  is that with the smallest  $\epsilon_b$  (saddle point) and will compete with the best path leading to cell  $A$ . So, we identify the block variables  $\epsilon_A$  and  $\epsilon_B$  with the saddle point of the best path leading to the corresponding cell, say

$$\epsilon_B = F_B [\{\epsilon_i\}_b] = \min_b \left[ \max_i \{\epsilon_i\}_b \right]. \quad (6)$$

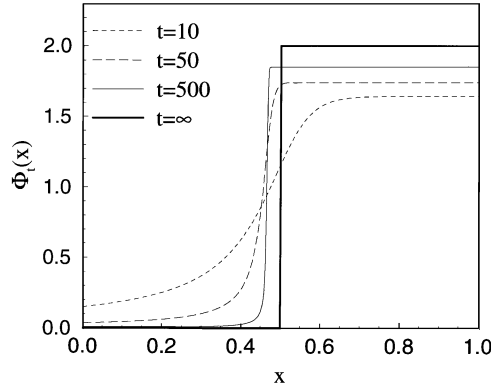


Fig. 1. – Time evolution of the solution of the equation for  $\Phi_t(x)$  (for 2 – d bond IP).  $\Phi_t(x)$  tends asymptotically to a theta-function with discontinuity at  $p_c = 1/2$ .

and analogously for  $\epsilon_A$ . Of course the initial density of the block variables  $\epsilon_A$  and  $\epsilon_B$  will be rather complex. However, extremal dynamics is not perturbed by a different choice of the initial density of the variables. In fact, eq. 2 is invariant under the transformation  $x \rightarrow \int_0^x p_0(y)dy$  which maps the density  $p_0(x)$  onto the flat one [6]. This is because we are interested in the relative probabilities of the various bonds and a change of the initial distribution for all bonds does not change these probabilities. So, we conclude that the coarse-grained dynamics is *intrinsically scale invariant*.

In order to use the FST method for the IP problem, we need the identification of *local growth rules*, that is the set of connected growth events which have the same statistical properties of the whole cluster. An avalanche is a *spatially and causally connected sequence* of single growth events. From the properties of the histogram one can see that a critical scale-invariant avalanche starts with a bond whose variable is equal to  $p_c$  [10], [3]. By definition, all other variables in the avalanche have values smaller than  $p_c$ . So [8] a generical critical avalanche has the same statistical properties of the whole cluster and in order to study the asymptotic cluster *we can consider only the dynamics inside such an avalanche*. In view of the above arguments, the RTS equations for this *local dynamics* are obtained from eqs. (2), (3) by taking into account

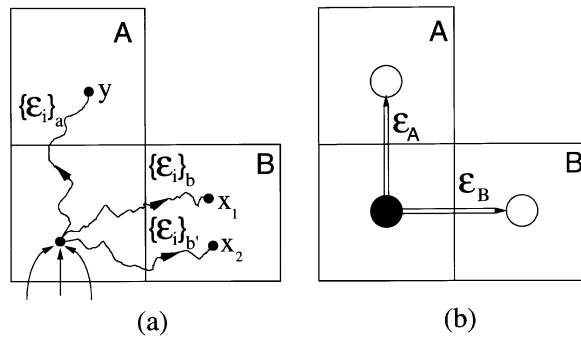


Fig. 2. – Renormalization scheme for the extremal dynamics: a) Dynamics at the smaller scale; b) rescaled dynamics.

only the variables which become active after the initiator's growth and by integrating in eq. (2) only in  $[0, p_c]$  and not in  $[0, 1]$ . This will lead to a GPD for the avalanche dynamics which is defecting, *i.e.* with a normalization less than one, in that it must account of the probability that the avalanche stops [8].

At this point, we use the scale-invariant IP local growth rule in the FST scheme, to compute the fractal dimension of the infinite percolating cluster [8]. This is accomplished by computing the transverse nearest-neighbour correlations at a generic scale via suitable path integrals. In table I we show the results of our FST calculations for invasion percolation and for some related models. The convergence of the path integrals with respect to the path length  $n$  is power law like [6], and we need to extrapolate the FST results to  $n = \infty$ . Our results are in very good agreement with the known values (table I).

We can compute also the exponent of the critical avalanche distribution of IP. The size-distribution of avalanches in IP ( $p \neq p_c$ ) has been shown to have the form [3], [12]:

$$D(s; p) = s^{-\tau} f(|p - p_c|s^\sigma), \tag{7}$$

where  $s$  is the avalanche size and  $p$  is the initiator of an avalanche. In the limit  $t \rightarrow \infty$  the system self-organizes into the critical state  $p = p_c$ , and the (normalized) avalanche size distribution becomes:

$$D(s; p_c) = \frac{s^{-\tau}}{\sum_{s=1}^{\infty} s^{-\tau}} \tag{8}$$

Our calculation scheme develops in the following steps: 1) we evaluate the left hand side of eq. (8); 2) we solve eq. (8) for  $\tau$ . For  $s = 1$  we can write

$$D(s = 1; p_c) = \sum_j W^{(j)} \cdot P^{(j)}(s = 1; p_c), \tag{9}$$

where  $W^{(j)}$  are the weights of the different boundary conditions near the initiator of the avalanche and  $P^{(j)}(s = 1; p_c)$  are the stopping probability of the avalanche after one step, computed with the local RTS growth rules.

TABLE I. – *FST results for IP without trapping ( $D_f$ ), with site trapping ( $D_f^1$ ), with bond trapping ( $D_f^{11}$ ), and directed IP ( $D_f^{\text{DIP}}$ ). The FST results are compared with known analytical and simulation values.*

Order $n$	$D_f(n)$	$D_f^1(n)$	$D_f^{11}(n)$	$D_f^{\text{DIP}}(n)$
3	1.7039	1.6965	1.7029	1.6254
4	1.7941	1.7378	1.7825	1.6626
5	1.8228	1.7506	1.8066	1.6924
6	1.8473	1.7599	1.8245	1.7081
7	1.8565	1.7642	1.8317	1.7189
8	1.8645	1.7678	1.8372	1.7250
9	1.8677	1.7697	...	...
...	...	...	...	...
$\infty$	1.8879	1.7812	1.8544	1.7444
analytical	$\frac{91}{48} \simeq 1.895^{(a)}$	--	--	$\simeq 1.748^{(b)}$
simulation	$\sim 1.89$ [2]	$\sim 1.82$ [10]	$\sim 1.86$ [10]	--

<sup>(a)</sup> Conformal mapping applied to  $2d$  Percolation [13].

<sup>(b)</sup> Series expansions [14].

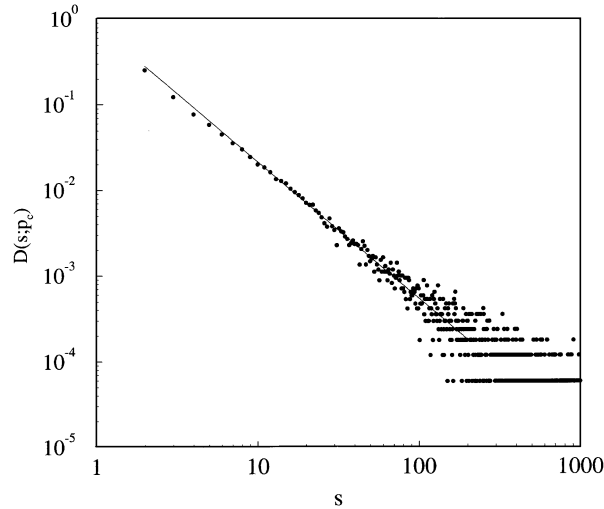


Fig. 3. – The distribution of critical avalanches in invasion percolation. The solid line has slope  $-1.60 \pm 0.02$ .

By inserting eq. (9) in eq. (8), we get:  $\tau \simeq 1.5832$ . Also this analytical result is in excellent agreement with numerical simulations [12], which give  $\tau = 1.60$ . In order to check independently our theory, we have performed a new type of computer simulations for the avalanche distribution. From the RTS scheme and the discussion of the scale-invariant local dynamics, one notes that  $k(t_0 + s)$ , the age of the bond grown during the avalanche at time  $t_0 + s$ , must satisfy the condition  $k(t_0 + s) < s$ . So, we can analyze the integer valued signal  $k(t)$ , instead of  $\epsilon(t)$  [10], [12] (the value of the smallest variable at time  $t$ ), in order to estimate numerically the exponent  $\tau$ . This method allows us to avoid the problems of the numerical approximations that one faces when one analyzes the real signal  $\epsilon(t)$ . We get  $\tau \simeq 1.60 \pm 0.02$  (fig. 3), in very good agreement with our analytical calculation. The knowledge of the exponents  $D_f$  and  $\tau$  one can recover, via scaling relations [12], to all the other critical exponents of IP.

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