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Invasion percolation and the time scaling behavior of a queuing model of human dynamics

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Abstract. In this paper we study the properties of the Barabási model of queuing under the hypothesis that the number of tasks is steadily growing in time. We map this model exactly onto an invasion percolation dynamics on a Cayley tree. This allows us to recover the correct waiting time distribution $P_W(\tau) \sim \tau^{-3/2}$ at the stationary state (as observed in different realistic data) and also to characterize it as a sequence of causally and geometrically connected bursts of activity. We also find that the approach to stationarity is very slow.

Keywords: percolation problems (theory), stochastic processes (theory), growth processes, diffusion

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1. Introduction

In this paper we present one particular model of queue evolution, based on the recent observations that in Web browsing, email communications and ordinary mail correspondence [1], the waiting time distribution (WTD) $P_W(\tau)$ of the time τ elapsed between a solicitation and the answer to it (e.g. receiving an email and answering it) often displays a fat tail and more particularly a power-law behavior. Queuing theory [2]–[4] describes a wide range of human dynamical behaviors [5, 6]. Most traditional models lead to exponential *waiting time distributions* $P_W(\tau) \sim \exp(-\tau/\tau_0)$ for the tasks in the queue. In contrast, some models recently introduced assign priorities to the tasks to be executed and in particular cases they generate power-law WTD $P_W(\tau) \sim \tau^{-\alpha}$ for the tasks in the queue.

In this paper we study a particular version of these latter models, namely the Barabási queuing model (BQM) [7, 8]. In our version of the BQM at each time step the task with highest random priority is always executed and replaced in the queue by a constant number $m \geq 2$ of new tasks with random priorities. This process can be mapped exactly onto an invasion percolation (IP) dynamics [9] on a Cayley tree [10], with a series of advantages. Firstly we can characterize the task list dynamics through the WTD at the stationary state. Secondly we show that its general evolution is composed of a sequence of geometrically and causally connected bursts of activities (task avalanches) with scale-invariant size distribution. Thirdly, we can study the dynamics out of stationarity and we show that the approach to it is very slow. Finally, it permits us to simply generalize the results to the case of time-varying m. As regards this latter point, one could think that a queue whose length increases on average in time is not sustainable for either humans or electronic devices. Nevertheless, in experimental data, for example the numbers of letters exchanged by Charles Darwin, not only are fluctuations in m present, but on average it is also possible to determine (see figure 1) a slow average increase of the queue length with time. This observation is thought to hold also for email communication and Web browsing. For instance in general we receive more emails than we answer, and the 'Inbox' length would increase fairly steadily if we did not erase some of the messages.

In the general BQM [7] one starts with an initial list (i.e. queue) of $n_0 \geq 2$ tasks. At every time step t one of these tasks is executed and replaced with $m(t)$ other new tasks.

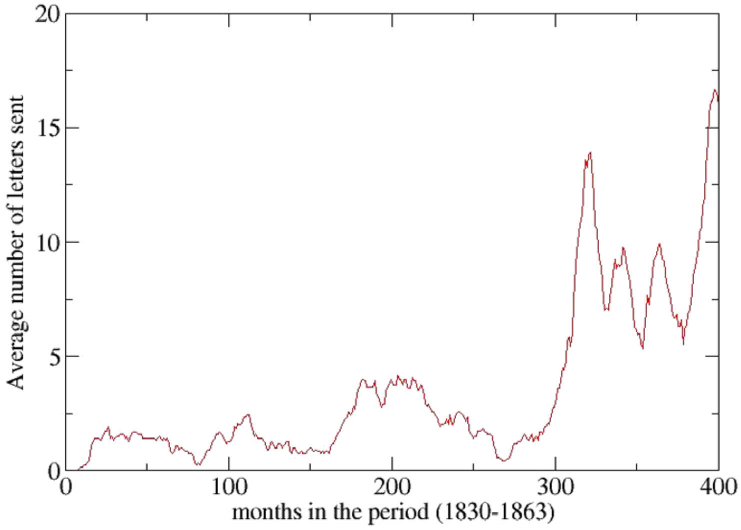

Figure 1. The time activity of letters sent by Charles Darwin.

For constant $m(t) = 1$, the queue length remains constant. The execution rule at each time step is given by fixing a random priority index $x_i \in [0,1]$ for each task in the queue and then executing with a probability $p \leq 1$ the task with the highest priority and with a probability $(1-p)$ a randomly chosen task. The related problem for general $0 \leq p \leq 1$ and $m = 1$ has been analyzed and solved in [11, 12]. In the purely extremal (i.e. when $p = 1$) case with a variable queue length, the behavior of $P_{\mathrm{W}}(\tau)$ differs strongly from the previous one. In [13] the case in which at each time step there is a probability $\mu \leq 1$ of executing the highest priority task while a new task is added to the list with another probability $\nu \leq 1$ has been studied. For $\mu = \nu = 1$ the above case of conserved queue length is recovered. If at least one of μ and ν is strictly smaller than 1, the list length instead varies in time. Depending on whether $\mu > \nu$, $\mu = \nu < 1$ or $\mu < \nu$ the WTD $P_{\mathrm{W}}(\tau)$ for the stationary state changes its asymptotic behavior.

For $\mu > \nu$ the stationary state is characterized by executing all tasks in the queue with a WTD such that $P_{\mathrm{W}}(\tau) \sim \tau^{-3/2} \exp(-\tau/\tau_0)$ for $\tau \ll \tau_0$ and $P_{\mathrm{W}}(\tau) \sim \tau^{-5/2} \exp(-\tau/\tau_0)$ for $\tau \gg \tau_0$ with $\tau_0 = 4\mu(1-\nu)/(\mu-\nu)^2$. However, for $\mu = \nu < 1$, all tasks are executed with $P_{\mathrm{W}}(\tau) \sim \tau^{-3/2}$ with no upper cut-off, while for $\mu < \nu$, the mean queue length grows linearly in time, and one can show that asymptotically for $t \to \infty$ all tasks with priority index $x_i < (1-\mu/\nu)$ are never executed, staying forever in the queue. However, tasks with $x_i \geq (1-\mu/\nu)$ are executed with a WTD coinciding with the one for $\mu = \nu < 1$.

2. The model and the mapping to invasion percolation

In our version of the model the most urgent task is executed with probability $\mu = 1$ and replaced in the list with a constant number $m \geq 2$ of new tasks with random priorities. All the features of this model can be clarified by mapping it to an IP dynamics on a Cayley tree. A similar mapping to a 1D IP has proved to be fruitful also in the case of fixed queue length [12]. Invasion percolation on a Cayley tree [10] is defined as follows (see figure 2(a)): let us take a Cayley tree with branching ratio m where initially only the

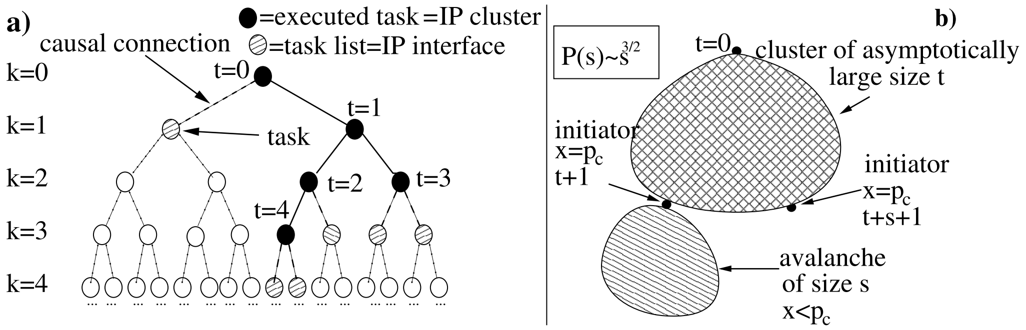

Figure 2. (a) Sketch of the first four steps of IP dynamics on a Cayley tree with branching ratio $m = 2$. (b) Illustration of a causally and geometrically connected avalanche whose size distribution is $P(s) \sim s^{-3/2}$. This characterizes the stationary state of both IP and the task queue dynamics.

top vertex site of the tree is occupied. A random number (*fitness*) x, extracted from a given probability density function (PDF) $p(x)$, is assigned once and forever to each empty site (independently of the others). At each time step the site of the growth interface $\partial\mathcal{C}_t$ with the highest fitness is occupied. $\partial\mathcal{C}_t$ is defined at each time t as the set of empty sites connected by a first-nearest-neighbor rule to the connected growing cluster $\mathcal{C}_t$ of occupied sites up to that time. Since for each occupied site m other new sites enter the growth interface, the numbers of sites in $\mathcal{C}_t$ and in $\partial\mathcal{C}_t$ at time t are respectively $\|\mathcal{C}_t\| = (t+1)$ and $\|\partial\mathcal{C}_t\| = m + (m-1)t$. The dynamics being extremal, the statistical and geometrical features of IP are independent of the shape of $p(x)$; our choice is to take $p(x) = 1$ with $x \in [0, 1]$.

The exact mapping between IP and our queuing model is by identifying sites with tasks, fitness with priority index, growth interface $\partial\mathcal{C}_t$ in IP with the task list (i.e. the queue), and finally the growing IP cluster $\mathcal{C}_t$ with the set of executed tasks up to time t.

3. Avalanche dynamics and the waiting time distribution (WTD)

For our purposes we focus on the following features of the asymptotic stationary state of IP dynamics:

(1) The distribution (also called the *normalized interface histogram*) $\phi_s(x)$ of the fitnesses of the interface sites (i.e. of the tasks in the queue) has the step-function shape

$$\phi_s(x) = p_c^{-1}\theta(p_c - x), \tag{1}$$

where $p_c = (m-1)/m$ is the ordinary percolation threshold of the Cayley tree. This implies that: (i) apart from a vanishing fraction (i.e. a finite number) of sites, all the interface sites have fitness $x < p_c$; (ii) since the number of sites in the stationary state is infinite, only those few sites with $x \geq p_c$ can grow at each time step. Indeed, at each time for the just occupied site (executed task) $m \geq 2$ new sites (new tasks) enter the interface (queue). This implies that a fraction $(m-1)/m$ of the sites entering the interface at any time will never be executed. Since the interface site with maximal fitness is always executed and the 'fresh' interface sites have random fitness, asymptotically only and all the sites with $x \geq (m-1)/m$ are executed while the others stay forever in the interface.

(2) The cluster of occupied sites substantially coincides with the incipient percolating cluster of ordinary percolation (i.e. at occupation probability $p = p_\mathrm{c}$).

(3) The stationary dynamics self-organizes into a sequence of spatially and causally connected avalanches of growth activity [14] with a scale-invariant size distribution $P(s) \sim s^{-3/2}$, independently of the value of m. Each of these avalanches (see figure 2(b)), say $\mathcal{A}$, starts with the growth of a site (the *initiator* of $\mathcal{A}$) with fitness $x = p_\mathrm{c}$ exactly, meaning that all the other interface sites at that time have $x \leq p_\mathrm{c}$. Following this growth, m new sites/tasks (*sons*), geometrically connected to the initiator, enter the interface. $\mathcal{A}$ stops immediately if all sons have fitness $x < p_\mathrm{c}$, and consequently another 'old' interface site grows with $x = p_\mathrm{c}$ (due to the shape of $\phi_s(x)$), therefore initiating a new avalanche $\mathcal{B}$, i.e., $\mathcal{A}$ lasted only one step. If instead at least one of the sons of the initiator of $\mathcal{A}$ has $x > p_\mathrm{c}$, then $\mathcal{A}$ goes on at least one step further as one of these sons grows. Consequently, m other new 'descendants' (sons of a son) of the initiator of $\mathcal{A}$ enter the interface. Again $\mathcal{A}$ continues if at least one among all the remaining descendants of any generation (called the *avalanche interface*) has fitness $x > p_\mathrm{c}$; otherwise the avalanche stops, and so on.

The exponent of $P(s)$ can be computed analytically by mapping the avalanche dynamics to a problem of first return of unbiased random walks. Let n_t be the number of sites (tasks) with $x \geq p_\mathrm{c} = (m-1)/m$ after the tth step of an avalanche (i.e., on its interface). Since after the growth of one site m new sites enter the interface, we have the following Markovian evolution for $n_t > 0$:

$$n_{t+1} = n_t + j - 1 \qquad \text{with probability } \binom{m}{j} p_\mathrm{c}^{m-j}(1-p_\mathrm{c})^j, \tag{2}$$

with $j = 0, 1, \ldots, m$. That is, n_t follows an ordinary random walk with independent steps. As $p_\mathrm{c} = (m-1)/m$, the average increment of n_t in one time step is zero (the *martingale* property). Therefore [15] the probability distribution of the time s for which $n_s = 0$ for the first time (i.e. the duration of the avalanche) scales as $s^{-3/2}$ at large s. A percolation argument can also be used to find the same exponent: as the avalanche initiator has $x = p_\mathrm{c}$ and the avalanche lasts exactly for a time interval equal to the number of sites with $x > p_\mathrm{c}$ connected to it in the positive time direction, the avalanche size is distributed like the finite clusters at the critical point $p = p_\mathrm{c}$ in ordinary percolation on the same tree: $P(s) \sim s^{-3/2}$.

Random walk and diffusion theory arguments also permit us to evaluate the stationary state WTD $P_\mathrm{W}(\tau)$ for the tasks with $x > p_\mathrm{c}$. We follow here a discussion similar to that of [13]. We can write the WTD as

$$P_\mathrm{W}(\tau) = \sum_{n=0}^{\infty} \int_{p_\mathrm{c}}^{1} \mathrm{d}x\, \tilde{Q}(n,x) G(n,x,\tau), \tag{3}$$

where $\tilde{Q}(n,x)$ is the probability that at a generic time step at the stationary state we have exactly n tasks in the queue (i.e. sites on the IP interface) with priority larger than $x \geq p_\mathrm{c}$. The quantity $G(n,x,\tau)$ is instead the conditional probability that, always at the stationary state, a certain task with priority $x \geq p_\mathrm{c}$ added to the list at a time step when

n other tasks with priority larger than x are present is executed after τ time steps. We can write the evolution equation for the number $n_t(x)$ of tasks in the list with priority larger than x at time t. We can write for $n_t(x) \geq 1$

$$n_{t+1}(x) = n_t(x) + j - 1 \qquad \text{with probability } \binom{m}{j} x^{m-j}(1-x)^j, \tag{4}$$

where $j = 0, 1, \ldots, m$, which is similar to equation (2). We can consequently write the master equation for the probability $Q(n, x, t)$ that at time t there are exactly n tasks in the list with priority larger than x. The aim of this simplification is to let us write it for $m = 2$, for which $p_c = 1/2$. From equation (4) we can write for $n \geq 3$

$$Q(n, x, t+1) = Q(n+1, x, t)x^2 + Q(n, x, t)2x(1-x) + Q(n-1, x, t)(1-x)^2, \tag{5}$$

while for $n \leq 2$ we have

$$\begin{aligned} Q(2, x, t+1) &= Q(3, x, t)x^2 + Q(2, x, t)2x(1-x) + Q(1, x, t)(1-x)^2 + Q(0, x, t)(1-x)^2, \\ Q(1, x, t+1) &= Q(2, x, t)x^2 + Q(1, x, t)2x(1-x) + Q(0, x, t)2x(1-x), \\ Q(0, x, t+1) &= Q(1, x, t)x^2 + Q(0, x, t)x^2. \end{aligned} \tag{6}$$

$\tilde{Q}(n, x)$ is given by the stationary solution of the above equations. In order to find both $\tilde{Q}(n, x)$ and $G(n, x, \tau)$ we can now proceed in a similar way to [13]. It is simple to show that the well-normalized stationary solution for $x \geq p_c = 1/2$ of equations (5) and (6) is

$$\begin{aligned} \tilde{Q}(n, x) &= \frac{2(x-p_c)}{x^2}\left[\frac{(1-x)^2}{x^2}\right]^{n-1}, \qquad \text{for } n \geq 2, \\ \tilde{Q}(1, x) &= 2\frac{1-x^2}{x^2}(x-p_c), \qquad \tilde{Q}(0, x) = 2(x-p_c). \end{aligned} \tag{7}$$

Note that for $x \to p_c^-$, each $\tilde{Q}(n, x) \to 0$ with the ratio $\tilde{Q}(n, x)/\tilde{Q}(l, x) \to 1$ for any $n, l \geq 2$, i.e., the distribution of the number $n_{t\to\infty}(p_c)$, becomes practically uniform.

The quantity $G(n, x, \tau)$ can be found using equation (5) in complete analogy with [13] or [16], leading to the same correct scaling behavior $P_W(\tau) \sim \tau^{-3/2}$. We refer here to the former, as it is of simpler formulation. First of all we note that equation (5), in both the continuous time and $n = y$ approximations, becomes the diffusion equation

$$\partial_t Q(y, x, t) = c(x)\partial_y^2 Q(y, x, t) + d(x)\partial_y Q(y, x, t), \tag{8}$$

with $c(x) = x^2$ and $d(x) = x^2 - (1-x)^2$. Since we are considering $x \geq p_c = 1/2$ we have $d(x) \geq 0$, i.e., there is a drift in the small y (i.e. n) direction. $G(n, x, \tau)$ can be seen as the probability that at the stationary state, at fixed x and given that at time $t = 0$ it is $y = n$, one has $y = 0$ for the first time at time $t = \tau$. This implies that [13, 15]

$$G(n, x, t) = -\partial_t\left[\int_0^\infty \mathrm{d}y\, Q(y, x, t)\right],$$

where here $Q(y, x, t)$ is the solution of equation (8) with initial condition $Q(y, x, 0) = \delta(y-n)$. All this gives

$$G(n, x, \tau) = \frac{n}{\sqrt{4\pi c(x)t}} \exp\left\{-\frac{[n-d(x)t]^2}{4c(x)t}\right\}.$$

We now use this result and equation (7) in equation (3) to find $P_W(\tau)$. It is simple [13] to show that for large τ we have $P_W(\tau) \sim \tau^{-3/2}$. In other words each task with $x \geq p_c$ has to wait a finite portion of the avalanche duration before being executed. Note that all these results are completely independent of the integer branching factor $m > 1$. From equation (4) it is natural to expect to have the same result in the case in which at each time step m is not constant but fluctuates with independent fluctuations such that $\langle m \rangle \geq 1$, and finite variance. This is why our model has the same statistical features as that in [13]. In the case where $\langle m^2 \rangle = +\infty$, we expect anomalous exponents for both $P(s)$ and $P_W(\tau)$ as the random walk equations (2) and (4) become Levy flights, as shown in [17]. As shown in section 1, the case in which $\langle m \rangle < 1$ (i.e. $m = 1$ with probability $p < 1$ and $m = 0$ with probability $1 - p$) has been exactly solved in the above cited paper [13] for the values $\mu = 1$ and $\nu < \mu$ of respectively the execution and entrance rates of the tasks in the queue.

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4. Finite time dynamics and approach to stationarity

We now address the question on how fast the approach to stationarity is in such models. Again some rigorous theoretical results for IP on a tree turn to be very useful to this end. We summarize here the main results from the literature, and then propose a simple mean-field approach showing how slow the relaxation to the right stationary state is. In [10] the main exact result, adapted to our notation, states that the probability that at time t of the dynamics a task with priority smaller than $(p_c - \epsilon)$ is executed vanishes exponentially fast for large t for $\epsilon > 0$, but as $t^{-1/2}$ for $\epsilon \to 0^+$. This suggests that deviations from the stationary dynamics due to stochastic fluctuations of the fitness statistics disappear in a single realizations as $t^{-1/2}$. Such a result can be confirmed by analyzing equation (4). As mentioned before, this equation says that the number of tasks with fitness larger than x follows an ordinary biased random walk. The average increment in a single time step is $\overline{\delta n}(x) = m(p_c - x)$ which is respectively positive and negative for $x < p_c$ and $x > p_c$. At the same time we have that the variance of this increment is $\Delta^2 = m(m-1)(1-x)^2 - m(p_c - x) - m^2(p_c - x)^2 > 0$. Therefore for $x < p_c$ there is a positive bias which permits us to apply the central limit theorem at large t, saying that for these values of x, $n_t(x)$ at large t is approximately a Gaussian variable with average value $\langle n_t(x) \rangle \simeq m(p_c - x)t$ and variance $\langle \Delta n_t^2(x) \rangle \simeq \Delta^2 t$. Since the total number of tasks in the queue at time t grows like $(m-1)t$ this permits us to say that the fitness distribution in the queue at large time t in a single dynamical realization is given by the average equation (1) plus random fluctuations of order $t^{-1/2}$. This coincides with the previous rigorous result[1].

All this concerns the fluctuations from the asymptotic stationary dynamics in a single realization at large t. We now present a simple mean-field argument to show that even the approach of quantities averaged over all realizations, i.e. where stochastic fluctuations are averaged out, shows a transition to the asymptotic stationary behavior as slow as t^{-1}. We study the dynamics of the above introduced normalized distribution $\phi(x, t)$ of the priorities of the tasks in the queue (fitness *histogram* of interface sites in IP) at time t averaged over all realizations. In order to write a closed equation for $\phi(x, t)$ we use the

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[1] The tasks with fitness $x > p_c$, due to the fact that the bias is negative and $n(x, t)$ cannot be negative, are always finite, and consequently contribute only with a random fluctuation of order t^{-1}.

run time statistics (RTS); this is a probabilistic method introduced for describing IP and related dynamics, and we evaluate the statistical weight of all different 'histories' of the dynamics (i.e. paths in the realization space) [18]–[20]. Let $h(x,t)\,\mathrm{d}x$ be the number of tasks in the queue at time t with priority in $[x, x+\mathrm{d}x]$ in a single realization. We can write rigorously

$$h(x,t+1) = h(x,t) - m(x,t+1) + m, \tag{9}$$

where $m(x,t)$ is the PDF of the priority of the executed task at time t conditional on the whole past history. By calling $p_i(x,t)$ the PDF of the priority of the ith task in the queue at time t conditional on the same past history, and assuming that the executed task at that time is the jth, a good approximation for $m(x,t+1)$ [19] is $m(x,t+1) = (1/\mu_j(t))p_j(x,t)\prod_{i(\neq j)}^{\partial\mathcal{C}_t}[\int_0^x \mathrm{d}y\, p_i(y,t)]$, where $\mu_j(t) = \int_0^1 \mathrm{d}x\, p_j(x,t)\prod_{i(\neq j)}^{\partial\mathcal{C}_t}[\int_0^x \mathrm{d}y\, p_i(y,t)]$ is the probability of selecting j conditional on the past history. We now average equation (9) over all the possible realizations of the dynamics up to time $(t+1)$ using the symbol $\langle\cdot\rangle_{t+1}$ for this average. By definition we have $\langle h(x,t+1)\rangle_{t+1} = (m-1)(t+1)\phi(x,t+1)$ and $\langle h(x,t)\rangle_{t+1} = \langle h(x,t)\rangle_t = (m-1)t\phi(x,t)$. In order to take the same average for $m(x,t+1)$ note that, if $A(i_0, i_1, \ldots, i_{t-1}, i_t)$ is a function of the queue history up to time $(t+1)$ identified by the sequence of executed tasks $\{i_0, i_1, \ldots, i_{t-1}, i_t\}$, by the rules of conditional probability, we can write

$$\langle A(i_0, i_1, \ldots, i_t)\rangle_{t+1} = \left\langle \sum_{j\in\partial\mathcal{C}_t} \mu_j(t) A(i_0, i_1, \ldots, i_{t-1}, j)\right\rangle_t .$$

We therefore have

$$\langle m(x,t+1)\rangle_{t+1} = \left\langle \sum_{j\in\partial\mathcal{C}_t} p_j(x,t) \prod_{i(\neq j)}^{\partial\mathcal{C}_t} \left[\int_0^x \mathrm{d}y\, p_i(y,t)\right]\right\rangle_t .$$

Considering that by definition $\langle p_j(x,t)\rangle_t = \phi(x,t)$, we now introduce the mean-field approximation consisting in replacing the average of the above products of $p_l(x,t)$ with the products of the averages, i.e.,

$$\langle m(x,t+1)\rangle_{t+1} = (m-1)t\phi(x,t)\left[\int_0^x \mathrm{d}y\, \phi(y,t)\right]^{(m-1)t-1} . \tag{10}$$

We can now write the mean-field equation for $\phi(x,t)$ as

$$\phi(x,t+1) = \frac{t}{t+1}\phi(x,t)\left\{1 - \left[\int_0^x \mathrm{d}y\, \phi(y,t)\right]^{(m-1)t-1}\right\} + \frac{m}{(m-1)(t+1)}. \tag{11}$$

This strong decorrelating approximation is expected to lead to a faster relaxation to stationarity than the actual one. We show however that, even in this approximation, the stationary state is the right one and the approach to it is a power law. Integrating both sides of equation (11), taking the continuous time limit and $t \gg 1$ we get

$$\partial_t\psi(x,t) = \frac{-1}{t+1}\left[\psi + \frac{1}{m-1}\psi^{(m-1)t} - \frac{mx}{m-1}\right], \tag{12}$$

where $\psi(x,t) = \int_0^x \mathrm{d}x' \, \phi(x',t)$ is the cumulative average priority distribution, and we have assumed $\psi(0,t) = 0$ at all t. The initial condition for equation (12) is $\psi(x,0) = x$. Since $\phi(x,t)$ is a normalized PDF, we have $\psi(x,t) \geq 0$, non-decreasing in x, and $\psi(1,t) = 1$.

In the x region where $(1-\psi) \gg 1/[(m-1)t]$ we can approximate equation (12) simply with

$$\partial_t \psi(x,t) = -\frac{1}{t+1}\left(\psi - \frac{mx}{m-1}\right), \tag{13}$$

which leads to the solution for sufficiently large t

$$\psi(x,t) = \frac{mx}{m-1}\left(1 - \frac{1}{mt}\right) \qquad \text{for } x < \frac{m-1}{m} - \frac{1}{m^2 t}. \tag{14}$$

Note that $p_c = (m-1)/m$. Moreover in the x region where $\epsilon = (1-\psi) \ll 1/[(m-1)t]$ it is simple to show that the following approximation holds:

$$\partial_t \epsilon(x,t) = -\epsilon(x,t) + \frac{m(1-x)}{(m-1)(t+1)}, \tag{15}$$

whose solution is $\psi(x,t) = 1 - \epsilon(x,t)$ with

$$\epsilon(x,t) = \frac{m(1-x)}{(m-1)(t+1)}\left[1 + \mathrm{O}\left(\frac{1}{t}\right)\right] \tag{16}$$

when $x \gg (m-1)/m$. All this means that

$$\phi(x,t) = \frac{\theta(p_c - x)}{p_c} + \delta\phi(x,t),$$

with $\delta\phi(x,t) \sim 1/t$. Therefore even in this mean-field approximation, for which we expect a faster relaxation, deviations from it vanish as slowly as $1/t$.

5. Conclusions

In conclusion, we have shown a way to analytically compute all the main features of the Barabási model of human dynamics with time-increasing queue length. This is done by using invasion percolation on a Cayley tree and random walk theory. We believe that the approach that we introduced allows us to describe quantitatively two intuitive features of task queues. The first feature is that some tasks seem to remain almost indefinitely before being processed; secondly we recover naturally the fact that in the real world execution of a task often has the effect of generating an avalanche of new tasks. Through our approach one can study both the stationary state dynamics and the approach to it. This shows that both are characterized by temporal power laws, as is typical for extremal dynamics in quenched disorder [18]–[20].

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