



# Non-perturbative renormalization group approach to surface growth

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## Abstract

We present a recently introduced real space renormalization group (RG) approach to the study of surface growth. The method permits us to obtain the properties of the KPZ strong coupling fixed point, which is not accessible to standard perturbative field theory approaches. Using this method, and with the aid of small Monte Carlo calculations for systems of linear size 2 and 4, we calculate the roughness exponent in dimensions up to  $d = 8$ . The results agree with the known numerical values with good accuracy. Furthermore, the method permits us to predict the absence of an upper critical dimension for KPZ contrarily to recent claims. The RG scheme is applied to other growth models in different universality classes and reproduces very well all the observed phenomenology and numerical results. Intended as a sort of finite size scaling method, the new scheme may simplify in some cases from a computational point of view the calculation of scaling exponents of growth processes. © 1999 Elsevier Science B.V. All rights reserved.

The study of the non-equilibrium dynamics of rough surfaces and interfaces is a topic of growing interest [1–3]. Many efforts have been devoted to single out the behaviors shared by apparently different growing phenomena. In this context the search of universality classes, permitting to categorize different models and systems of rough surfaces sharing the same scaling properties, is a central task. The main milestone in this direction is the Kardar–Parisi–Zhang equation (KPZ), which is the minimal Langevin equation capturing the essential physics of many different growing surfaces beyond the Gaussian linear (Edward–Wilkinson) theory [1–3]. It reads as [4]

$$\frac{\partial h(x, t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, t), \quad (1)$$

where  $h(x, t)$  is a height variable at time  $t$  and position  $x$  in a  $d$ -dimensional substrate of linear size  $L$ .  $\nu$  and  $\lambda$  are constants and  $\eta$  is a Gaussian white noise. The KPZ equation has become a paradigm of surface growth processes and an overwhelming amount of both numerical and analytical studies have been performed since it was first proposed. It is also related to other physical systems as: directed polymers in random media [2], systems with multiplicative noise [5], and the Burgers equation [6].

A central quantity of interest in the study of surfaces is the roughness  $W(L, t)$ , defined as the fluctuation of the height variable around its mean value,  $\bar{h}(t)$ ; that is:

$$W^2(L, t) = \frac{1}{L^d} \sum_x [h(x, t) - \bar{h}(t)]^2. \quad (2)$$

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In many seemingly unrelated problems the large scale behavior of the roughness is observed to be scale invariant, in the sense that it grows in time as  $W(L, t) \sim t^\beta$  until it saturates to a stationary value at a characteristic time,  $t_s \sim L^z$ , with universal exponents  $\beta$  and  $z$ . Once in the saturated regime, the roughness obeys  $W(L) \sim L^\alpha$ . These critical exponents are not independent, in fact, from a trivial scaling argument  $\alpha = \beta z$ , and as a consequence of a tilting invariance [6,1,2]  $\alpha + z = 2$ . There is therefore only one independent exponent, say  $\alpha$ .

Standard field theoretical RG studies of the KPZ equation predict the presence of a *roughening transition* above  $d = 2$  [7]; i.e. there are two RG attractive fixed points: A trivial (Gaussian) one with  $\alpha = 0$  which describes a flat phase, and one describing the rough phase. This last one is *non-perturbative*, therefore standard perturbative methods fail to give any prediction for the exponents in the rough phase. One fundamental question, presently under debate, is to elucidate if there is a finite upper critical dimension [8]. Summing up, any new systematic approach to the problem of surface growth would be certainly desirable.

In this paper we review a recently introduced real-space, non-perturbative renormalization group scheme [9–11] permitting to determine roughness exponents in growth processes (even in strong coupling regimes) with the aid of relatively small Monte Carlo computations. This method can be seen as a finite size scaling method, permitting to determine from the analysis of very small system sizes (which is very convenient from a computational point of view) scaling exponents with reasonable accuracy, and constitutes a powerful tool in the analysis of growth processes.

Let us now describe the method as applied to a microscopic model in the KPZ universality class [9]: The solid-on-solid (SOS) model (see [1–3] for the definition and details). Instead of studying its microscopic dynamics defined over cells of unit size  $L_0 = 1$ , and height  $h_0$  in a substrate of total length  $L$ , we describe the model at a coarse-grained level [9]. For that purpose we first define blocks or cells, of linear size  $L_k = 2^k$  and height  $h_k = h_0(2^k)^\alpha$ .  $\alpha$  is for the time being an unknown exponent to be determined latter in a self-consistent way. A system configuration is specified by giving the heights  $h(i)$  at scale  $k$ , at every substrate site,  $i$  ( $i = 1, \dots, b$ , where  $b = L/2^k$ ). Two configura-

tions are considered equivalent if they differ by a constant height at every site. Having defined the blocks, now we introduce an effective coarse-grained dynamics, or growth rules at a generic scale. This is done by giving the transition rates from a generic surface configuration to a new one with one additional grown block. The transition rates should contain the relevant information of the microscopic dynamics. The growing rate at site  $i$  is defined as

$$P[h(i) \rightarrow h(i) + h_k] \equiv L_k + \lambda_k \sum_{j \text{ nni}} \max[0, h(j) - h(i)]. \quad (3)$$

The term  $L_k$  is the contribution of the vertical growth (i.e. random deposition) and the sum over neighbor block sites  $j$  accounts for the lateral growth. We hypothesize that these are the two only possible types of growth in the dynamics of the SOS coarse-grained dynamics. The ratio  $x_k = \lambda_k/L_k$  represents the relative importance of the lateral growth from an elementary lateral step in comparison with the growth through random deposition at scale  $k$ . The parameter  $x_k$ , necessary to fully specify the dynamics, is to be determined self-consistently by imposing scale invariance.

Iterating in time the aforementioned growth rules, a statistically stationary state should be reached, the averaged width of which could be determined as a function of  $L_k$ ,  $x_k$  and  $h_k$ . At this point we can pose ourselves the following question: given a description of the system at scale  $k$ , defined by parameters  $(L_k, h_k, x_k)$ , what will be the values of  $h_{k+1}$  and  $x_{k+1}$  at the scale  $L_{k+1}$ ? To answer this question we consider a given system of size  $L = L_{k+2}$  and describe it at two different levels of coarse graining: using  $b$  blocks of size  $L_{k+1} = L_{k+2}/b$  and then dividing each component block of size  $L_{k+1}$  again in  $b$  blocks of size  $L_k$ , or alternatively considering directly  $b^2$  blocks of size  $L_k = L_{k+1}/b = L_{k+2}/b^2$ . The condition that the total roughness takes the same value when calculated using blocks of two different scales gives a renormalization equation in a way to be explained in what follows. In particular, we use the following a property of the roughness,  $W^2$ : Partitioning the system in cells of size  $a$ , one obtains  $W^2(L) = W^2(a) + W^2(L/a)$ , i.e. the total roughness,  $W^2(L)$ , is equal to the roughness of the

system measured using cells of size  $a$ ,  $W^2(L/a)$ , plus the internal roughness of one of these cells,  $W^2(a)$ . First we apply the last equation to the case for which  $L = L_{k+1}$  and  $a = L_k = L_{k+1}/b$ , obtaining:  $W^2(L_{k+1}) = W^2(L_k) + W^2(b; x_k)$  where  $W^2(b; x_k)$  is the stationary roughness for  $b$  blocks for  $x_k$ , which is proportional to  $h_k^2$ , and therefore can be formally written as  $W^2(b; x_k) = R(x_k, b)h_k^2$ . As shown in [9]  $h_k^2 = cW^2(L_k)$  where  $c$  is a constant [12]; this simply reflects the fact that the height  $h_k$  at scale  $k$  has to be taken proportional to the roughness at that scale. Combining the last two results, we can write

$$W^2(L_{k+1}) = W^2(L_k)(1 + cR(x_k, b)). \quad (4)$$

Applying twice this relation we can relate scales  $k+2$  and  $k$ ,

$$W^2(L_{k+2}) = W^2(L_k)(1 + cR(x_{k+1}, b))(1 + cR(x_k, b)). \quad (5)$$

Another possibility, as said previously, is to relate directly scales  $k+2$  and  $k$ , applying once the relation (4) replacing  $b$  by  $b^2$ . This leads to:

$$W^2(L_{k+2}) = W^2(L_k)(1 + cR(x_k, b^2)). \quad (6)$$

Equating the r.h.s. of Eqs. (5) and (6) we get the renormalization equation:

$$(1 + cR(x_k, b^2)) = (1 + cR(x_{k+1}, b))(1 + cR(x_k, b)) \quad (7)$$

which gives, once the function  $R$  is known, the flow of the renormalized parameter  $x_k$  under changes of scale, and therefore permits one to evaluate the fixed point value  $x^*$  that characterizes the scale invariant dynamics. Knowing  $x^*$ , it is a straightforward task to determine the roughness exponent  $\alpha$ :

$$\alpha = \lim_{k \rightarrow \infty} \frac{1}{2} \log_2 \left( \frac{W^2(L_{k+1}, x^*)}{W^2(L_k, x^*)} \right) = \frac{1}{2} \log_2 (1 + cR(x^*, b)). \quad (8)$$

The previously exposed method was applied in [9] to study KPZ growth in finite dimensions (from  $d = 1$  to  $d = 8$ ) by analyzing the case  $b = 2$ . The saturated width (i.e. the function  $R$ ) was evaluated through Monte Carlo simulations of the effective dynamics in these small cells (namely  $2^d$  and  $4^d$ ) for different values of  $x$ . In all dimensions a stable fixed point

Table 1

Results of the renormalization group calculation (first row) compared with numerical results (second row)

$d$	1	2	3	4	5	6	7	8
$\alpha_{\text{RG}}$	0.502	0.360	0.284	0.238	0.205	0.182	0.162	0.150
$\alpha_{\text{num}}$	0.5	0.387	0.305	0.261	0.193	0.18	0.15	–

is found with the associated values of  $\alpha$  reported in Table 1.

Observe the good agreement between numerical results obtained after extensive simulations [13], and our estimations using the new RG scheme with small cells [9]. These results support both the validity of the numerical simulations [13] and of our RG method. In particular, for dimensions larger than  $d = 4$  exponents differing from the mean field prediction  $\alpha = 0$  are observed, providing evidence that  $d = 4$  is not the upper critical dimension of KPZ.

The method can be applied to study analytically the infinite dimension limit [10]. For that, one just has to evaluate the width by using some theoretical argument. Assuming that the lateral growth rate is very large for large dimensions (assumption that can be verified a posteriori), it is a good approximation (as discussed in [10]) to consider only surfaces with two layers: high and low respectively. A state of the system can be characterized by the number of sites in high (low) position, and a master equation giving the flow between different system states can be written down. Calculating analytically the width of its associated stationary state with the previous hypothesis for two different cell sizes, it is a matter of simple algebra to obtain a lower bound for  $\alpha$  for large  $d$  [10]

$$\alpha \geq \frac{1}{3(\ln 2)^2} \frac{1}{d} \quad (9)$$

and the associated fixed point is stable for all  $d$ . This result excludes the possibility of having a finite upper critical dimension for KPZ-like processes.

Even though we have presented the method as a scheme to analyze KPZ-like processes the underlying ideas are quite general and can in principle be applied to other growth processes. In particular we have studied linear growth models (i.e. KPZ with  $\lambda = 0$ ), that should belong in the free (Gaussian) universality class, and verified that, as expected [3], the non-trivial fixed point seems to disappear for  $d \geq 2$ . We have also applied the method to other highly non-trivial growth

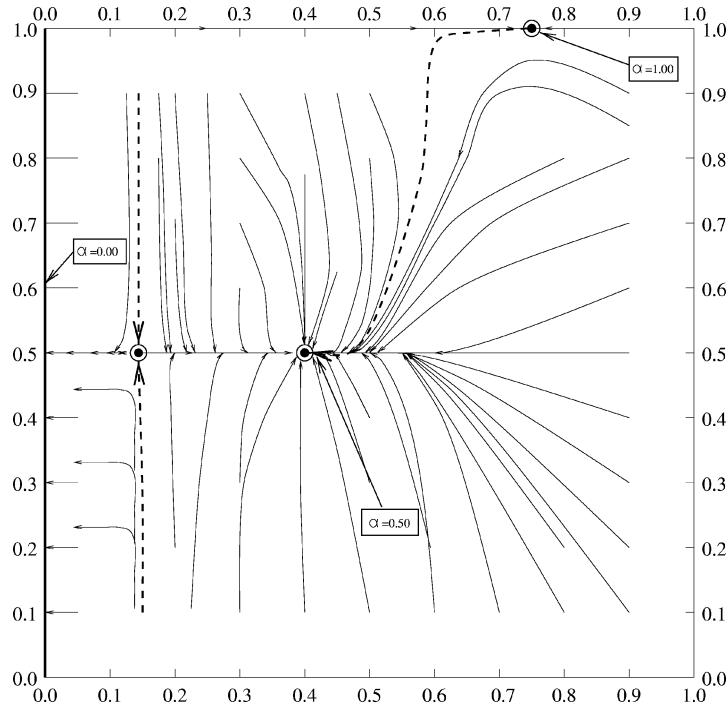


Fig. 1. RG flow for the two parametric model described in the text.

models that exhibit roughening transitions even in one dimension [14]. In particular we have constructed a two-parameter representation of the model presented in [14]: with probability  $p$  a particle is added at a randomly chosen site, while with complementary probability  $1 - p$  the value of the height at that site is changed by averaging it with its nearest neighbors weighted with a parameter  $a$  [11]. This model exhibits a rich phenomenology as the parameters,  $a$  and  $p$  are varied [11]. There are three different scaling regimes:

- (i)  $\alpha = 1$  for  $a = 1$ ,
- (ii) for  $a < 1$  and small values of  $p$  there is a smooth phase with  $\alpha = 0$ , and
- (iii) for  $a < 1$  and larger values of  $p$ ,  $\alpha = 0.5$ .

This complex behavior is perfectly reproduced when the model is analyzed by the RG scheme. In particular in Fig. 1, we show the RG flow lines in the two-dimensional parameter space showing the presence of different fixed points with the expected values of  $\alpha$ , and a separatrix signaling the roughening transition.

All the previous results support the validity of the new non-perturbative real space RG scheme as an

useful tool to analyze growing surfaces. The method itself can be considered as a sort of finite-size scaling procedure, and provides therefore a computational tool to facilitate the computation of scaling exponents.

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