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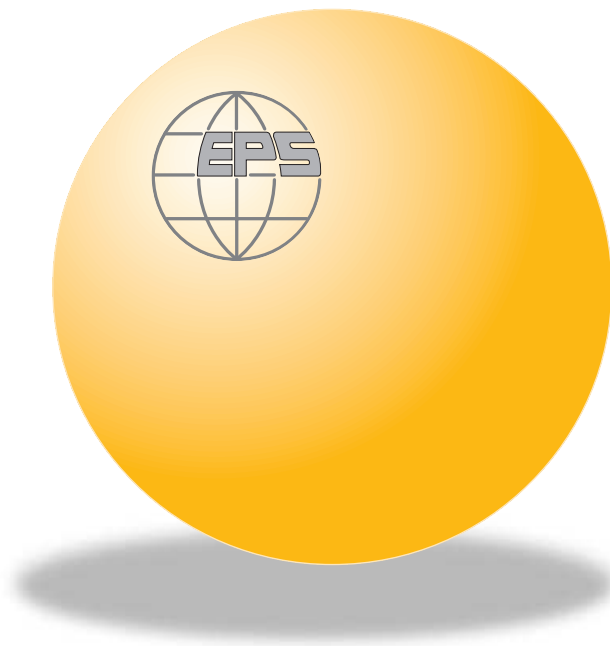
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A study with exactly solvable one-dimensional models**

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## Causality constraints on fluctuations in cosmology: A study with exactly solvable one-dimensional models

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**Abstract.** – A well-known argument in cosmology gives that the power spectrum (or structure function)  $P(k)$  of mass density fluctuations produced from a uniform initial state by physics which is causal (*i.e.* moves matter and momentum only up to a finite scale) has the behaviour  $P(k) \propto k^4$  at small  $k$ . Noting the assumption of analyticity at  $k = 0$  of  $P(k)$  in the standard derivation of this result, we introduce a class of solvable one-dimensional models which allows us to study the relation between the behaviour of  $P(k)$  at small  $k$  and the properties of the probability distribution  $f(l)$  for the spatial extent  $l$  of mass and momentum-conserving fluctuations. We find that the  $k^4$  behaviour is obtained in the case that the first *six* moments of  $f(l)$  are finite. Interestingly, the condition that the fluctuations be localised —taken to correspond to the convergence of the first two moments of  $f(l)$ — imposes only the weaker constraint  $P(k) \propto k^n$  with  $n$  anywhere in the range  $0 < n \leq 4$ . We interpret this result to suggest that the causality bound will be loosened in this way if quantum fluctuations are permitted.

In cosmology “causality bounds” are very important in various contexts. They are limits which can be inferred simply from the fact, which is an intrinsic feature of Big Bang models, that there is a finite horizon for causal processes, *i.e.* light can travel only a finite distance in the time since the Big Bang. One important example is an argument, due originally to Zeldovich [1], which gives a strong constraint on the power spectrum (*i.e.* what is usually called the structure function in statistical physics) describing mass fluctuations. It states that if fluctuations are built, starting from a uniform distribution of matter, by causal physics (*i.e.* physical processes moving matter and momentum coherently up to a maximal scale), then the small- $k$  form of the power spectrum is  $P(k) \propto k^4$ . The importance of this argument is in its corollaries: it implies that the spectrum considered to correctly describe the perturbations at

very large scales observed [2] in the microwave background,  $P(k) \sim k$ , cannot be produced by causal physics acting prior to the time when radiation decouples from matter. And it is one of the motivations for, and successes of, the popular “inflation” model that it can produce such fluctuations (by modifying the causal structure of the Big Bang model at early times). In this letter we describe a study of this bound —for which there is no rigorous demonstration— from a purely statistical physics perspective: causality is simply a bound on the distance over which matter can be moved coherently in building a fluctuating mass distribution from an initially uniform background. We introduce a set of exactly solvable one-dimensional models which allow us to study how the  $k^4$  result depends on the precise constraint which is assumed to be imposed on real-space fluctuations. Our central result is that the condition of localisation of the fluctuations (in the sense usual in statistical physics, *i.e.* finite first and second moment) leads only to the weaker result  $P(k) \propto k^n$  with  $0 < n \leq 4$ , with the  $k^4$  result requiring additional constraints on higher moments. In the context of cosmology this suggests that the  $k^4$  bound should not be taken as valid when quantum fluctuations are taken into account.

Let us consider first a simple derivation of this  $k^4$  bound on fluctuations<sup>(1)</sup>. The power spectrum  $P(\vec{k})$  is defined as  $P(\vec{k}) = \lim_{V \rightarrow \infty} (\langle |\delta_\rho(\vec{k})|^2 \rangle / V)$ , where

$$\delta_\rho(\vec{k}) = \int_V d^3x e^{-i\vec{k}\cdot\vec{x}} \delta_\rho(\vec{x})$$

and  $\delta_\rho(\vec{x}) = (\rho(\vec{x}) - \rho_0)/\rho_0$  is the density fluctuation field around the mean density  $\rho_0$ . Assuming statistical homogeneity,  $P(\vec{k})$  is also given by the Fourier transform of the two-point correlation function  $\xi(\vec{x}) = \langle \delta_\rho(\vec{x}_0) \delta_\rho(\vec{x}_0 + \vec{x}) \rangle$ . Expanding  $P(\vec{k})$ , in powers of  $\vec{k}$ , one obtains, assuming statistical isotropy,

$$P(\vec{k}) \equiv P(k) = \int \xi(x) d^3x - \frac{k^2}{3!} \int x^2 \xi(x) d^3x + \frac{k^4}{5!} \int x^4 \xi(x) d^3x + O(k^6), \quad (1)$$

where the integrals are now over all space,  $k = |\vec{k}|$  and  $x = |\vec{x}|$ . The  $k^4$  result is obtained if the zeroth and second moments of  $\xi(x)$  vanish. Substituting a Taylor expansion of  $\delta_\rho(\vec{x})$ <sup>(2)</sup> to obtain  $P(k)$ , it is easy to show that these two conditions are equivalent to the requirement

$$P(0) = \lim_{V \rightarrow \infty} \frac{\langle (\Delta M)^2(V) \rangle}{V} = 0, \quad (2)$$

$$P''(0) \propto \lim_{V \rightarrow \infty} \frac{\langle \int_V (\vec{x} - \vec{y})^2 \delta_\rho(\vec{x}) \delta_\rho(\vec{y}) d^3x d^3y \rangle}{V} = 0. \quad (3)$$

That the first of these conditions, eq. (2), follows as a consequence of mass conservation and the causality constraint can be understood simply, for it is just equivalent to the condition that the fluctuations are sub-Poissonian, *i.e.* with a squared mass variance which grows less rapidly than the volume. Poisson fluctuations are an upper limit because, in order to obtain them from an initially uniform distribution, one clearly needs to move mass randomly over an

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<sup>(1)</sup>We note that condensed matter also provides examples of systems where the structure factor shows this small- $k$  behaviour. In particular, when a system is quenched from a homogeneous (high-temperature) phase into a broken-symmetry (low-temperature) phase, the kinetics of the system is characterized by order growth [3], and the structure factor associated to the field describing the order has at late stages of this process a scaling expression  $S(k, t) \sim k_m^{-3} F(k/k_m(t))$ , where  $k_m$  corresponds to the maximum of  $S$  at a given time  $t$ . For systems whose phase ordering is characterized by a scalar field which is conserved during the process, the scaling function  $F(x)$  goes as  $x^4$  [4, 5].

<sup>(2)</sup>In cosmology, the standard derivation of the bound (see, *e.g.*, [6–9]) uses directly such a Taylor expansion of  $\delta_\rho(\vec{x})$ , which is problematic as the coefficients in this expansion are formally divergent.

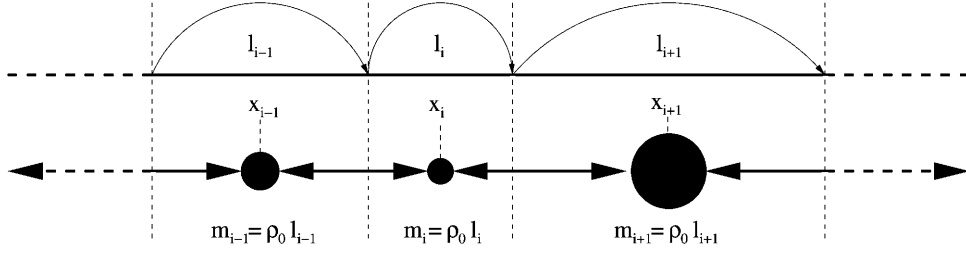


Fig. 1 – A schematic representation of our one-dimensional models. Subsequent uncorrelated steps of length  $l$  with probability distribution  $f(l)$  define a fragmentation of the line. The mass in each segment is then aggregated to a point in its centre. Mass and centre of mass are thus explicitly conserved locally.

average scale proportional to the size of the system. The causality constraint, however, places an upper bound on this scale.

The link between the second condition, eq. (3), and momentum conservation is less evident. Indeed, the integral is not just proportional to the variance in the momentum given by  $\langle \int_V \vec{x} \cdot \vec{y} \delta_\rho(\vec{x}) \delta_\rho(\vec{y}) d^3x d^3y \rangle$ , which, in fact, diverges faster than the integration volume  $V$ . The necessary finiteness of the coefficient of  $k^2$  (if it is finite) is ensured by the cancelling pieces which come from the coefficients of the constant and  $k^2$ -term in the expansion of  $\delta_\rho(\vec{k})$ , *i.e.* from the zeroth and *second* moments of the fluctuation field [5].

Putting aside this remaining subtlety in the demonstration of the  $k^4$  result, we concentrate here on another aspect of it: the implicit assumption which has been made that the Taylor expansion of  $P(k)$  at  $k = 0$  is well defined (at least to order  $k^4$ ). What is the content of this assumption, and is it justified here? Fourier transform theory tells us that if  $\xi(x)$  is rapidly decreasing at large  $x$ , then its Fourier transform  $P(k)$  is analytic at  $k = 0$ . In cosmology, causality might be—and often is—taken to require that physical quantities are strictly uncorrelated beyond the horizon scale, *i.e.* that any correlation function is *identically* zero beyond this scale. In this case, the assumption of analyticity is thus justified. Such a constraint on the correlation function is, however, wholly classical. In quantum field theory causality is imposed as a constraint on the commutators of operators at space-like separations [10], and does not require that correlation functions vanish. Our approach in this letter—through the study of the models we will now describe—is to define the constraint from causality as one on the form of the permitted fluctuations in real space, and then to determine what this constraint implies about the behaviour of  $P(k)$  as  $k \rightarrow 0$ .

So let us consider now the following one-dimensional toy model for the generation of fluctuations (see fig. 1). We start with a continuous exactly uniform mass density  $\rho_0$ . Choosing randomly a first point, we take successive uncorrelated steps to the right of length  $l$ , where  $l$  is a positive number chosen randomly at each step according to the probability density  $f(l)$ . Doing the same on the left, we define a division of the real line into segments. We then gather up the mass in each segment into a point, placing it in the *centre* of the segment. This defines a mass- and momentum-conserving discretisation of the initial mass density. The properties with respect to causality in this model depend wholly on those of the probability function  $f(l)$ , as  $l/2$  is the maximum distance through which mass and momentum must be moved to create the fluctuations. We will return after our analysis of the model to discuss the subtleties of this constraint. Note that the discreteness of the distribution is not important here. By applying a smearing (*e.g.*, Gaussian) window function, one could map the distribution onto a

continuous one. Such a smearing affects only the large- $k$  properties of  $P(k)$ , and thus changes nothing as the results below.

We now derive exact expressions for the power spectrum in this model, and then study the small- $k$  behaviour which interests us. The mass density in the segment  $[0, L)$  of finite length  $L$  is

$$\rho(x) = \rho_0 \sum_{m=1}^N l_m \delta(x - x_m), \quad (4)$$

where  $x_i = \sum_{j=1}^{i-1} l_j + l_i/2$  and the  $l_i$  are the lengths of the  $N$  consecutive segments defining points in  $[0, L)$ . The finiteness of the first and second moments of  $f(l)$  allows us to apply the central-limit theorem to infer that the number of points  $N$  in such a segment is Gaussian-distributed in the large- $L$  limit about the mean  $\bar{N} = L/\langle l \rangle$  with fluctuations  $\delta N \sim \sqrt{L}$ . The mean density is then defined, *i.e.* we have  $\langle \rho(\bar{x}) \rangle = \rho_0$ . From its definition it is easy to show that the power spectrum for  $k \neq 0$  is given as  $P(k) = \lim_{L \rightarrow \infty} P_L(k)$ , where

$$P_L(k) = \frac{1}{L} \left\langle \left| \sum_{m=1}^N l_m e^{-ikx_m} \right|^2 \right\rangle \quad (5)$$

which can be written as

$$P_L(k) = \frac{1}{L} \left\langle \sum_{i=1}^N l_i^2 \right\rangle + \frac{2}{L} \text{Re} \left( \left\langle \sum_{i < j} l_i l_j e^{-ik(x_j - x_i)} \right\rangle \right). \quad (6)$$

Since we are interested in the limit  $L \rightarrow \infty$ , we can evaluate the ensemble average with the number of points equal to the mean value  $\bar{N}$ . The first ensemble average gives  $\bar{N} \langle l^2 \rangle$ , while the second, by using the definition of  $x_i$ , can be written as

$$\langle l e^{-ikl/2} \rangle^2 \sum_{j=2}^{\bar{N}} \sum_{i=1}^{j-1} (\langle e^{-ikl} \rangle)^{j-i-1}, \quad (7)$$

where  $\langle g(l) \rangle = \int_0^\infty dl g(l) f(l)$  for any function  $g(l)$ . Performing the sum we find the exact expression for the power spectrum in the limit  $L \rightarrow +\infty$ :

$$P(k) = \frac{1}{\langle l \rangle} \left[ \langle l^2 \rangle - 2 \text{Re} e \frac{(\tilde{f}'(\frac{k}{2}))^2}{1 - \tilde{f}(k)} \right], \quad (8)$$

where  $\tilde{f}(k) = \int_0^\infty e^{-ikl} f(l) dl$  is the characteristic function for the probability distribution, and  $\tilde{f}'(k)$  its first derivative.

We can now discriminate various different cases. First, let us suppose that  $\tilde{f}(k)$  is an analytic function at  $k = 0$ . All the moments of the probability distribution then exist and one can write the Taylor expansion

$$\tilde{f}(k) = \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} k^j \langle l^j \rangle. \quad (9)$$

Substituted in eq. (8) this gives, to leading order at small  $k$ :

$$P(k) = \frac{k^4}{576} \left( \frac{\langle l^2 \rangle \langle l^3 \rangle^2}{\langle l \rangle^3} + \frac{\langle l^6 \rangle}{\langle l \rangle} - 2 \frac{\langle l^3 \rangle \langle l^4 \rangle}{\langle l \rangle^2} \right) + \text{O}(k^6), \quad (10)$$

reproducing the standard  $k^4$  result, with a  $P(k)$  which is analytic at  $k = 0$ .

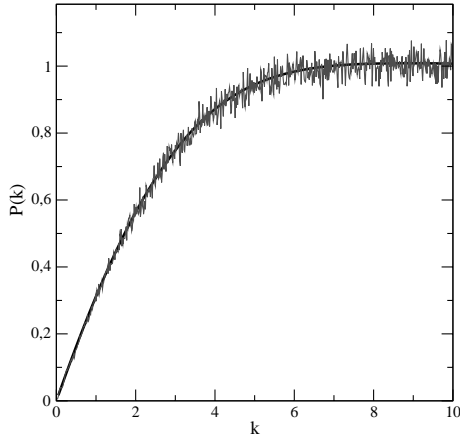


Fig. 2 – The power spectrum measured in a simulation of our one-dimensional model with  $f(l) = (1 + l^2)^{-2}$  (*i.e.*  $\alpha = 3$ ) for a box size  $L = 400$ , and one thousand realisations. The black line is the exact result obtained directly from eq. (8). Both curves show the predicted small- $k$  behaviour  $P(k) \propto k$ .

The derivation of eq. (8) required, however, only the convergence of the first two moments. Let us suppose that  $f(l)$  is such that only its first  $n$  ( $\geq 2$ ) moments are finite, *i.e.* that  $f(l) \sim l_o^\alpha / l^{\alpha+1}$  for large  $l$ , with  $\alpha > 2$  (and therefore  $n$  the integer part of  $\alpha$ ). In this case the small- $k$  behaviour of  $\tilde{f}(k)$  is given by

$$\tilde{f}(k) = \sum_{j=0}^n \frac{(-i)^j}{j!} k^j \langle l^j \rangle + Ak^\alpha l_o^\alpha + O(k^{n+1}) \quad (11)$$

for non-integer  $\alpha$ , where

$$A = \int_0^\infty \frac{du}{u^{\alpha+1}} \left( e^{-iu} - \sum_{j=0}^n \frac{(-i)^j}{j!} u^j \right). \quad (12)$$

For integer  $\alpha$  there are logarithmic corrections to these formulae. Substituting eq. (11) in eq. (8) we find the leading small- $k$  behaviour

$$P(k) = C_\alpha \frac{l_o^\alpha}{\langle l \rangle} k^{\alpha-2} + O(k^{\alpha-1}) \quad (13)$$

for  $2 < n \leq 6$ , where  $C_\alpha$  is a (positive) dimensionless constant proportional to the real part of  $A$ . For  $\infty > n > 6$ , when at least the first six moments are defined, eq. (10) is obtained again, but with higher-order corrections which are non-analytic. The non-analyticity of  $P(k)$  at  $k = 0$  implies that the correlation function  $\xi(x)$ , which in the case that all moments of  $f(l)$  are finite is rapidly decaying at large separations, shows instead an algebraic tail ( $\xi(x) \propto 1/x^{\alpha-1}$ ).

While it is the asymptotic behaviour at small  $k$  which we concentrate on here, the expression eq. (8) gives an exact expression for the power spectrum for all  $k$ . In fig. 2 we show both the result of such an exact calculation for the case  $f(l) = (1 + l^2)^{-2}$  (*i.e.*  $\alpha = 3$ ), and a numerical simulation of the same model. The expected small- $k$  behaviour  $P(k) \propto k$  can be clearly observed.

Let us consider the interpretation of these results. We have been able to derive the small- $k$  behaviour of the power spectrum when the first two moments of  $f(l)$  are taken to be finite. This requirement of the fluctuations corresponds simply to the standard definition of localised fluctuations [11]. They are short-range correlations with a well-defined characteristic scale, in contrast to long-range correlations (and so-called “broad” distributions [11]) which lack such a scale. Thus we have imposed on the fluctuations that they are effectively cut-off at a finite length scale, which naturally then explains the bound obtained: as noted earlier, the condition  $P(0) = 0$  corresponds exactly to the requirement that fluctuations in mass in real space should be sub-Poissonian. When we move mass coherently only up to a finite scale, we expect Poissonian fluctuations to form an upper bound on what one can obtain. Note that this is also a result which we expect to be true independently of the spatial dimension, as the relation between the exponents of the power spectrum and the classification (see [12] for a discussion) of mass fluctuations in Poissonian, sub-Poissonian and super-Poissonian is independent of dimension<sup>(3)</sup>. Note also that the correlation function, as we have seen, decays at least as fast as a power law with exponent greater than the dimension ( $\alpha > 2$ ), which indeed corresponds to an effectively short-range correlation with an integrable correlation function (whose integral is zero).

While we have this simple interpretation of our results, what we have found is actually very non-trivial and surprising in another respect: the exponent at small  $k$ , which characterises the large-scale fluctuations, *is not universal* for this class of localised distributions with finite mean and variance. Rather the exponent depends in a subtle way on the convergence of higher moments. We stress that this means the small- $k$  behaviour is thus determined by the large masses (*i.e.* the tails) of these distributions despite the fact that they are not broad.

Now let us return to the interpretation of these results in relation to the cosmological causality bound, which was our original motivation. The problem of the causality bound can be addressed with this simple model, because  $l/2$  is the maximum distance through which mass and momentum must be moved to create the fluctuations. If we impose that  $f(l)$  is *strictly zero* above some finite scale  $l_c$ , no mass can be moved over a scale larger than this ( $l_c$  is then the “size of the causal horizon”). Such a constraint is what one would take to be imposed classically by causality. And our model gives in this case the  $k^4$  result, in accord with the standard cosmological bound. However, our model shows that a generalisation of this result to the case where the fluctuations are cut off at a characteristic scale  $l_c$ , but smoothly, is non-trivial. And it is precisely such a generalisation which is needed if the  $k^4$  bound is to be extended to the case in which quantum fluctuations are admitted (*i.e.* that the physics generating the fluctuations is quantum in nature). This is so because quantum correlation functions generically do not vanish at any separation, as the required sharp localisation of states is impossible (cf. Reeh-Schlieder theorem [14]). This means that to extend the standard result also to include quantum fluctuations, one must assume that any (causal) quantum-mechanical evolution giving rise to fluctuations in an expanding FRW Universe (with sub-luminal, non-inflationary expansion) gives rise to, for example, a strictly rapid decay of correlation functions at super-horizon scales, and excludes the rapid power law decay admitted by a generic definition of localisation of fluctuations. While a rapid decay as an exponential is indeed typical of purely quantum fluctuations in the simplest cases one can envisage (*e.g.* the quantum correlations at space-like separations of a free massive scalar field in its ground state), such an assumption is too restrictive a form of the condition that quantum fluctuations be localised. An interesting analogy is given by screening of electrostatic forces in a neutral plasma, which is analogous to the effect of the existence of a causal horizon relative to a theory in flat space, as it leads to a physical cut-off scale beyond which the effect of a source is no longer felt. While such

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<sup>(3)</sup>A generalization of this model to arbitrary dimensions will be presented elsewhere [13].

screening gives rise to exponential decay of the (charge-charge) correlation function in the case of a Coulomb potential, it can be shown easily [15] that, for a  $1/r^2$  potential, the same physical mechanism leads to an effective short-range potential which decays algebraically (as  $-1/r^4$  in three dimensions). Indeed, the spectrum of fluctuations (at thermal equilibrium) at small  $k$  in this case behaves as  $P(k) \propto k$ , instead of  $P(k) \propto k^2$  in the case of electrostatics [15]. This suggests that a power spectrum of fluctuations  $P(k) \propto k$  at small  $k$  might, for example, be obtained from an appropriate interacting field quantized in a (sub-luminally expanding) FRW background. Such a model would provide an interesting alternative to the standard (non-causal) inflationary mechanism of producing such fluctuations at superhorizon scales.

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