

# Shaping large Poisson Voronoi cells in two dimensions

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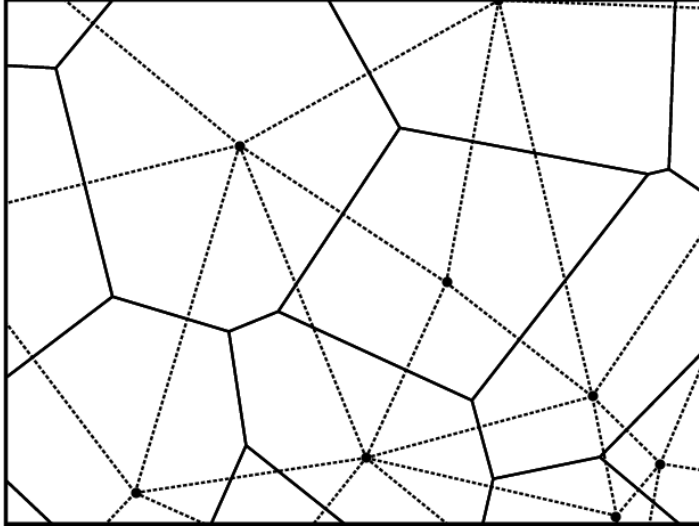
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Space tessellation in cells of different polyhedral forms is a widespread pattern observed in many natural contexts ranging from physics to biology and others [1]. Given a set of points, called *centers* or *seeds*, in a  $d$ -dimensional space, Voronoi tessellation is a particular case of space tessellation in which the space is partitioned into polyhedral regions, called Voronoi cells, each one defined to be the set of spatial points closer to a given seed than to any other one. For instance for  $d = 2$  each cell is a convex polygon. The particular case of Voronoi tessellation when the seeds are randomly distributed in space (i.e. defining a Poisson point process) is called Poisson Voronoi tessellation (PVT) (see figure 1). It is the most investigated space partitioning for different reasons: (i) it is the simplest model of disordered tessellation; (ii) many natural space partitionings into polyhedral cells, such as biological tissues, microemulsions and soap froths, are well approximated by a PVT. The applications of the PVT are extended also to more theoretical subjects such as the construction of random lattices in quantum field theory or the study of conduction and percolation in continuous media.

Many efforts have been made in the last fifty years by mathematicians and physicists to derive the main statistical properties of PVTs in different spatial dimensions with the aim of clarifying their geometrical aspects and to relate them to physical applications. In spite of the simplicity of the definition of the PVT, the derivation of most such statistical properties is, at least for  $d \geq 2$ , a very difficult task requiring subtle mathematical techniques and calculations.

The two most prominent quantities studied in PVT are (i) the probability  $p_n$  that a cell is  $n$ -sided and (ii) the probability  $g(v; \rho_0) dv$  that an arbitrary cell has a volume



**Figure 1.** Example of PVT for  $d = 2$ . The set of seeds form a Poisson point process and the cell corresponding to a seed  $i$  is the intersection of all the half-planes of points closer to the seed  $i$  than to any other one.

$v \leq V < v + dv$ , both for a Poisson distribution of seeds with an average number density (intensity)  $\rho_0$ . Both problems are exactly solved in one dimension for which we have respectively  $p_n = \delta_{n,2}$  (where  $\delta_{a,b}$  is the Kronecker delta) and

$$g(v; \rho_0) \equiv \rho_0 f(\rho_0 v) = 4\rho_0^2 v e^{-2\rho_0 v}.$$

While the first property is trivial, the derivation of the second one is based on two facts: (i) in a Poisson distribution of points the probability that a segment of size  $l$  is empty is  $e^{-\rho_0 l}$ , (ii) the occupation states of any two non-overlapping segments are statistically independent events. In higher dimensions the exact formulae for  $p_n$  and  $g(v; \rho_0)$  are not yet exactly known and different ansätze have been formulated that can be compared with the results of large numerical simulations. The main ansatz in the literature for the cell size distribution [2] is

$$f(x) = c \frac{b^{a/c}}{\Gamma(a/c)} x^{a-1} \exp(-bx^c),$$

where  $x = v/\langle v \rangle$  with  $\langle v \rangle = \rho_0^{-1}$ , and  $\Gamma(y)$  is the usual Euler Gamma function. For  $d = 2$  the largest available numerical simulations give a best fit with  $a \simeq 3.31$ ,  $b \simeq 3.04$  and  $c \simeq 1.08$ , while for  $d = 3$  one has  $a \simeq 4.81$ ,  $b \simeq 4.06$  and  $c \simeq 1.16$ .

The probability  $p_n$  is clearly independent of  $\rho_0$ , as in a Poisson point process  $\rho_0$  fixes the only characteristic scale, i.e., the length unit. Though a complete formula for  $p_n$  is unknown, some important properties of it for  $d = 2$  are that  $\langle n \rangle = 6$  in the limit of an infinite number of seeds, and that  $p_n$  decays to zero very fast at larger  $n$ . The exact asymptotic large  $n$  behavior of  $p_n$ , always for  $d = 2$ , has been also provided in [3]. This derivation required very technical arguments in some way related to the solution of the statistical mechanics partition function of a system of many interacting particles. Indeed it has been shown that  $p_n$  can be written as a  $2n$ -fold integral in the polar coordinates of the mid-points of the sides of the cell (i.e. the point of intersection between the side of

the cell and the segment connecting the seed of the cell to its neighbor sharing that side). The result is

$$p_n = \frac{C}{4\pi^2} \frac{(8\pi^2)^n}{(2n)!} [1 + \mathcal{O}(n^{-1/2})] \quad \text{for } n \rightarrow \infty, \quad (1)$$

or alternatively

$$\log p_n = -2n \log n + n \log(2\pi^2 e^2) - \frac{1}{2} \log(2^6 \pi^5 C^{-2} n) + \mathcal{O}(n^{-1/2}), \quad (2)$$

where

$$C = \prod_{q=1}^{\infty} \left(1 - \frac{1}{q^2} + \frac{4}{q^4}\right)^{-1}. \quad (3)$$

Another important result achieved with the same techniques is that for  $n \rightarrow \infty$  on average the cell shape becomes circular with a diverging radius

$$R_c = \left(\frac{n}{4\pi\rho_0}\right)^{1/2} \quad (4)$$

and finite (i.e. asymptotically  $n$ -independent) random local deformations around this distance. More subtle results can be found in [3].

It is however of great importance to provide also simple heuristic arguments for deriving to a good approximation the main statistical properties of the PVT. These approximate derivations, in contrast to difficult calculations based on special properties of the model studied, can potentially be of great value for the many experimental scientists working on systems showing Poisson Voronoi-like patterns. In addition they may suggest ways for finding relations with similar problems in different scientific fields. Such simple arguments have been put forward by Hilhorst in [4] where the dominant terms in equation (2) are derived through much simpler geometrical and informational arguments.

In a few words, the derivation of the first, dominant term of equation (2) is, under the hypotheses of a circular cell of radius  $R$  and sidedness  $n$ , based on the analysis of the available space for one of the mid-points (or equivalently for one first-neighbor seed) once the positions of the other  $(n-1)$  first-neighbor seeds are fixed. This leads to an available area of tangential size  $l \sim Rn^{-1}$  and thickness  $w \sim Rn^{-2}$ . While the first length scale is quite trivial to interpret as the fraction of the circumference per mid-point, the second one gives the non-trivial thickness of the circular annulus in which the mid-points of the  $n$  sides of the central cell are contained. The calculation then proceeds on writing  $p_n = \exp(\Delta S)$  where  $\Delta S$  is the entropy difference between the constrained configuration, having the  $n$  mid-points of the cell perimeter in the circular annulus of radius  $R$  and thickness  $w$  and all the other higher order neighbor seeds at a distance larger than  $2R$  from the seed of the central cell, and the unconstrained one. Such entropy difference is then maximized in  $R$ . This maximization gives directly the dominant term in equation (2) plus sub-dominant terms which are of the same form as the ones in the exact equation, but with undetermined coefficients. Moreover from this entropy maximization procedure, one finds that the typical radius of the cell is given by equation (4) and consequently that the typical thickness of the circular annulus of the cell mid-points is  $w \sim \rho^{-1/2} n^{-3/2}$ . The fact that the thickness  $w$  vanishes in the large  $n$  limit implies that asymptotically the

mid-points of a cell form an almost continuous and smooth curve. In order to go beyond this first-order approximation and to derive the second dominant term in equation (2), one has to perform a more subtle heuristic calculation, however again much simpler than the complete derivation of equation (2) given in [3]. It is based on approximating the  $2n$ -fold exact integral defining  $p_n$ , in such a way as to consider only circular cells and independent angles defining the sides of the cell perimeter. Finally, by studying fluctuations around this circular approximation, one can deduce that:

- (i) in the large  $n$  limit the shape of the cell can deviate from a pure circular form by long wavelength oscillations of finite (i.e.  $n$ -independent) amplitude (negligible in the large  $n$  limit as  $R \sim n^{1/2}$ );
- (ii) the number  $\nu_n^{\text{full}}$  of so-called Gabriel neighbors of a cell increases with  $n$  as  $\nu_n^{\text{full}} \sim n^{1/2}$  (a Gabriel neighbor is defined as a neighboring cell satisfying the condition that the straight line connecting the seed points of the two neighbors intersects their common border).

Together these results constitute the basic information needed to characterize the geometry and statistics of large  $n$ -sided cells in the PVT for  $d = 2$ . In summary we can conclude that the heuristic arguments developed in [4] give an important intuitive and physical basis for fundamental statistical properties of PVT which find important applications in different scientific fields. Moreover they can be the starting point for further developments in the field.

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