



## Complexity in cosmic structures

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### Abstract

We discuss correlation properties of a general mass density field introducing a classification of structures based on their complexity. Standard cosmological models for primordial mass fluctuations are characterized by a sort of large-scale *stochastic* order, that we call *super-homogeneity* to highlight the fact that mass fluctuations increase as a function of scale in the slowest-possible way for any stochastic mass field. On the other hand, the galaxy spatial distribution show complex structures with a high degree of inhomogeneity and fractal-like spatial correlations up to some relevant cosmological scale. The theoretical problem of cosmological structure formation should then explain the growth of strongly correlated and non-linear structures from the very uniform field of density fluctuations given as standard initial condition.

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Cosmic structures represent a very interesting playground for the methods of statistical physics and self-organization of complex structures. There are two main areas. The first deals with the very irregular spatial structures developed by the clustering of galaxies. The second broad area is provided by the cosmic microwave background radiation (CMBR) which is extremely smooth apart from very small-amplitude fluctuations. A general theory should link these two observations which appear quite different. Many new data for both areas are now available and much more are expected in the near future, creating big expectations, interest and animating challenges. Indeed, on the

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observational side, the data in cosmology have been growing exponentially in the last ten years, and in the coming decade a huge amount of new data will be available, in particular for the three-dimensional observations (via redshift) of galaxy distribution and CMBR anisotropies. Cosmology, therefore, is based more on observational and testable grounds than ever in the past. On the other hand, modern statistical physics can be able to provide a new general framework for the understanding of cosmic structures, both from the phenomenological and the theoretical points of view. It is important to notice that this new approach includes, as a particular case, the old analytical liquid-like approach (homogeneous mass density fields with small fluctuations), but it is also able to shed light in more complex cases as matter distribution in the universe seems to be. Here we briefly illustrate the main points discussing the properties of galaxy data and standard cosmological models (see Ref. [1] for more details).

We firstly introduce the basic properties of those stochastic systems (e.g. mass density fields) that can be represented as stationary stochastic process (SSP), where *stationary* refers to invariance under translation of the spatial statistical properties. The single realization of such a SSP can be thought to be a particular density field. In the context of cosmology the requirement of statistical stationarity and isotropy is justified by the fact that in the study of the mass distribution in the universe the cosmological principle is assumed: there are no preferential points or directions in the universe. Clearly, this principle must be intended in the statistical sense. This implies that the statistical properties of the mass distribution inside a sample volume should not depend on the location of the sample in the universe and on its spatial orientation: such a condition can be satisfied in both distributions with positive or zero ensemble average density (i.e., homogeneous or fractal).

Let us consider the stochastic mass distribution represented by the microscopic mass density function  $\rho(\vec{r})$  and focus on the case in which it is a discrete particle distribution: The integral of  $\rho(\vec{r})$  over an arbitrary volume gives the number of particles in such a volume. Considering particles of identical unitary mass this is also the mass contained in the volume. We may then write  $\rho(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i)$ , where  $\vec{r}_i$  is the position vector of the particle  $i$  of the distribution and  $\delta(\vec{x})$  is the Dirac delta function. In this context, the meaning of *homogeneity* (or spatial uniformity) in terms of the spatial average in a single realization of a stochastic mass distribution can be expressed as follows. Due to the usual assumption of ergodicity of the stochastic mass field, for a single realization of the mass distribution (i.e., a strictly non-negative field) the existence of a well-defined average positive density implies that [1] in  $d$  dimensions

$$\lim_{R \rightarrow \infty} \frac{1}{\|S(R; \vec{x}_0)\|} \int_{S(R; \vec{x}_0)} \rho(\vec{r}) d^d r = \rho_0 > 0 \quad \forall \vec{x}_0, \quad (1)$$

where  $\|S(R; \vec{x}_0)\| \sim R^d$  is the volume of the sphere  $S(R; \vec{x}_0)$  of radius  $R$ , centered on an arbitrary point  $\vec{x}_0$ .

A large-scale homogeneous stochastic mass distribution is in general characterized by the presence of structured (i.e., correlated) fluctuations around the average density, which can be of different nature depending on the ensemble statistical properties of the field. In order to characterize spatially these structures it is convenient to use the two-point correlation function. For example, one may consider the *reduced* two-point

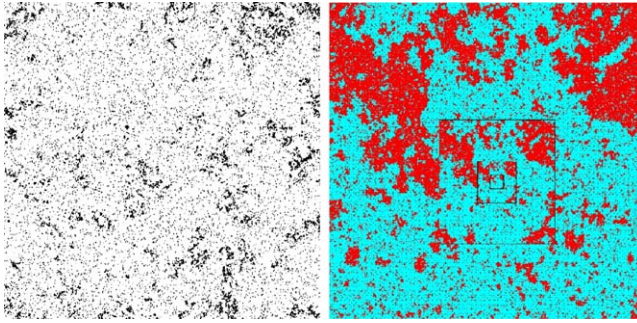


Fig. 1. Left panel: Substantially Poisson system: Statistically stationary and isotropic particle distribution with positive small-scale correlations. Right panel: Homogeneous critical system: the system is characterized by a positive background density with large-scale and power law correlations of density fluctuations (in the picture positive fluctuations are dark and negative ones are clear) There are clusters of fluctuations of a fixed sign of all sizes and the correlation length diverges  $r_c \rightarrow \infty$ . The absence of an intrinsic characteristic scale is shown by the fact that at different “zooms” the system looks the same, i.e., the fluctuations field (and not the whole field) is self-similar and has *fractal* features.

correlation function defined as  $C_2(r_{12}) = \langle (\rho(\vec{r}_1) - \rho_0)(\rho(\vec{r}_2) - \rho_0) \rangle$  (where  $\langle \cdot \rangle$  is the ensemble average symbol and  $r_{12} = |\vec{r}_1 - \vec{r}_2|$ ), which is the main function used to study and characterize spatial correlations between fluctuations around the positive average value. From the above definitions we see that  $C_2(r_{12})$  measures the spatial *memory* of mass density fluctuations on the scale  $r_{12}$ . In order to characterize through a single number the persistence of correlations in the fluctuation field, the concept of *correlation length*  $r_c$  has been introduced, for example, by

$$r_c^2 = \frac{\int d^d r r^2 C_2(r)}{\int d^d r C_2(r)}.$$

The concept of correlation length is useful to distinguish two important cases of large  $r$  behavior of  $C_2(r)$  (see Fig. 1): (i)  $C_2(r) \sim r^{-\gamma}$  with  $0 < \gamma < d$ , and (ii)  $C_2(r) \sim \exp(-r/r_c)$  ( $r_c$  is a measure of the correlation length) at large enough  $r$ . Case (i) is typical of a thermodynamic system (e.g. the mass density of a fluid at the liquid–gas critical point) at the point of a second-order phase transition and the main feature is the presence of fluctuation structures of all sizes (i.e., the fluctuation field around the positive average has fractal features), while (ii) is the ordinary behavior of it far from criticality: for  $r \gg r_c$  we have uncorrelated Poisson-like fluctuations (and for this reason the system is called *substantially Poisson*).

While the correlation length is the standard tool used to classify long- or short-range correlations in homogeneous stochastic-density fields, it is not suitable to detect the presence of a sort of long-range order. To this aim one has to focus on the large scale behavior of integrated mass fluctuations. Let  $M(R) = \int_{S(R)} \rho(\vec{r}) d^d r$  be the stochastic mass included in the sphere  $S(R)$  of radius  $R$ . Fluctuations of this quantity are measured by its variance  $\Sigma^2(R) = \langle M(R)^2 \rangle - \langle M(R) \rangle^2$ . One can show that for a homogeneous mass density field with a well-defined positive average  $\rho_0$ , the statistical counterpart of the homogeneity condition (Eq. (1)) is  $\lim_{R \rightarrow \infty} \Sigma^2(R) / \langle M(R) \rangle^2 = 0$ . Distributions

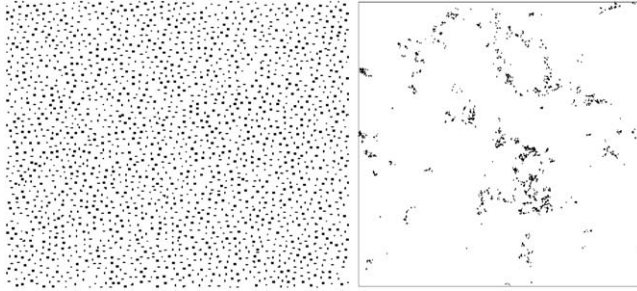


Fig. 2. Left panel: Super-homogeneous configuration: This is a realization of the One-Component Plasma. This is a projection of thin slices of three-dimensional distributions (from Ref. [2]). Right Panel: Fractal distribution with  $D=1.47$  in the two-dimensional Euclidean space. In this case there are structures and voids of all sizes.

(i) with infinite correlation length (critical systems) have  $\Sigma^2(R) \sim R^{d+\gamma}$  (with  $0 < \gamma < d$ ). Systems (ii) with finite correlation length (i.e., substantially Poisson) present  $\Sigma^2(R) \sim R^d$ . Finally, there is a special case of distribution characterized (iii) by a sort of long-range order (super-homogeneous) (see Fig. 2) in the density fluctuations for which  $\Sigma^2(R)$  increases even more slowly than case (ii), i.e.,  $\Sigma^2(R) \sim R^{d-\alpha}$  with  $0 < \alpha \leq 1$ . The case  $\alpha = 1$  is the limiting behavior for any stochastic mass distribution and it is due to a precise balance between positive and negative spatial correlations somewhat similar to what happens in an ordered lattice particle distribution.

Fractal geometry has allowed us to classify and study a large variety of structures in nature which are intrinsically irregular and self-similar [3]. Let us limit the discussion to statistically stationary and isotropic mass fields. The so-called *fractal* dimension  $D$  is the most important concept introduced to describe these intrinsically irregular systems. Basically, it measures the large-scale average logarithmic rate of increase of the “mass”  $\langle M(R) \rangle_p$  around an arbitrary point of the system (for this reason it is said *conditional* mass) with the size  $R$  of the volume in which it is measured: i.e.,  $\langle M(R) \rangle_p \sim R^D$ . In any case  $0 < D \leq d$ .  $D = d$  is found in all previous homogeneous systems where  $\rho_0 > 0$  is well defined. For  $D < d$  the conditional mass density seen in average by a point of the system goes to zero in a slow power law way, i.e., the system is *asymptotically empty*. Note that, however,  $D < d$  together with the spatial statistical stationarity of fluctuations imply that the mass field is wildly fluctuating at all scales. The concept of fractal dimension is important even for large scale homogeneous systems (e.g. possibly the galaxy distribution) to characterize *locally* the small scales region of strong clustering where the average density fluctuations are much larger than  $\rho_0 > 0$ . The scale beyond which fluctuations are smaller than  $\rho_0$  is called *homogeneity scale* and marks the cross-over between the *locally* fractal and the homogeneous behaviors.

Summarizing, we can frame stationary stochastic mass fields in two classes: (H) homogeneous (i.e.,  $\rho_0 > 0$ ) or (F) fractal (asymptotically empty). In the (H) class, the less asymptotically fluctuating systems are the super-homogeneous ones which are characterized by a sort of long-range order [2]; then larger fluctuations are found in the *substantially Poisson* systems with only small scale mainly positive correlations. Still

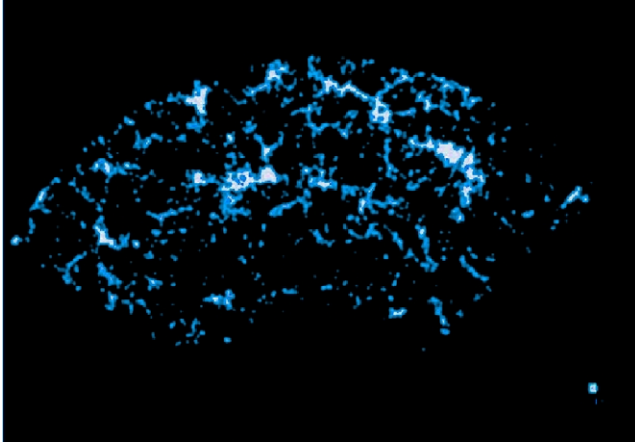


Fig. 3. This is a two-dimensional slice of thickness 5 Mpc of a volume limited sample extracted from the first data release of the Sloan Digital Sky Survey. The dimension of this sample is  $600 \times 300$  Mpc/h. The color scheme is proportional to the local density of galaxies. In this case there is no bias neither due to luminosity selection effects, nor to orthogonal projection distortions: Structures of galaxies of hundreds Mpc are still visible. The small square at the bottom right has a side of 5 Mpc, the typical clustering length according to the standard approach to the description of galaxy correlations.

larger fluctuations are found in homogeneous critical systems characterized by slow power-law correlations and a positive average density. Finally, we have the class (F) of self-similar mass distributions which are intrinsically inhomogeneous at all scales (the conditional average density decreasing as a slow power-law with the scale).

In order to connect this analysis to cosmology we have to consider that from one side there are the striking observations of the three-dimensional galaxy distribution which have shown a network of large-scale irregular clusters, filaments and voids (see Fig. 3). Actually on small but relevant scales (at least up to 20 Mpc/h) there is a general agreement that galaxy structures show fractal-like large fluctuations with power-law correlations [4,5]. On the other hand we find that all standard cosmological models compatible with Friedmann metric (e.g. the cold and the hot dark matter models) predict that the cosmological density field becomes super-homogeneous (1) (i.e., in view of the previous discussion, belonging to the class of less fluctuating stochastic mass distributions) on scales  $r \gtrsim 100$  Mpc/h. Indeed such a feature should be supported by observations of CMBR anisotropies [6]: particularly the large angular scale anisotropies should trace the super-homogeneous part of the distribution (1). The main theoretical problem is how to relate the super-homogeneous density field observed in the CMBR to a fractal-like distribution of galaxies and clusters on smaller scales. These are two different observations which have to be put together in a coherent theoretical framework. In this context, the central question concerns the extension of the fractal behavior: how large are the homogeneity scale and the largest non-linear (or strongly clustered) structures in the universe? This question introduces the next one: how can such large-scale structures of tens or even hundreds of Mpc have formed in the Hubble

time  $T_H$  (time scale of the universe)? The typical peculiar velocity of a single galaxy is about 500 km/s and in a time  $T_H \sim 15$  Gy it has traveled, in average, for few Mpc. Thus, we would expect not to see significant strong clustering on scales larger than 10 Mpc [7]. The explanation of the formation of these structures represent the main challenging problem in modern cosmology.

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