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BIASING IN GAUSSIAN RANDOM FIELDS AND GALAXY CORRELATIONS

ANDREA GABRIELLI,[1,2] FRANCESCO SYLOS LABINI,[2,3] AND RUTH DURRER[3]
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ABSTRACT

In this Letter, we show that in a Gaussian random field, the correlation length—the typical size of correlated structures—does not change with biasing. We interpret the amplification of the correlation functions of subsets identified by different thresholds as being caused by the increasing sparseness of peaks over threshold. This clarifies a long-standing misconception in the literature. We also argue that this effect does not explain the observed increase of the amplitude of the correlation function $\xi(r)$ when galaxies of brighter luminosity or galaxy clusters of increasing richness are considered.

Subject headings: galaxies: general — galaxies: statistics — large-scale structure of universe

We first explain, in mathematical terms, the notion of biasing for a Gaussian random field. Here we follow the ideas of Kaiser (1984; developed further in Bardeen et al. 1986). We then calculate biasing for some examples and clarify the physical meaning of bias in the context of Kaiser (1984). Finally, we comment on the significance of our findings for the correlations of galaxies and clusters.

We consider a homogeneous, isotropic, and correlated continuous Gaussian random field $\delta(\boldsymbol{x})$, with mean zero and variance $\sigma^2 = \langle\delta(\boldsymbol{x})^2\rangle$ in a volume V. The application of the following discussion to a discrete set of points is straightforward considering the effect of a smoothing length. The marginal one-point probability density function of δ is

$$P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\delta^2/2\sigma^2}.$$

Using P, we calculate the fraction of the volume V with $\delta(\boldsymbol{x}) \geq \nu\sigma$, $P_1(\nu) = \int_{\nu\sigma}^{\infty} P(\delta) d\delta$.

The correlation function between two values of $\delta(\boldsymbol{x})$ in two points separated by a distance r is given by $\xi(r) = \langle\delta(\boldsymbol{x})\delta(\boldsymbol{x} + r\boldsymbol{n})\rangle$. By definition, $\xi(0) = \sigma^2$. In this context, homogeneity means that the variance σ^2 and the correlation function $\xi(r)$ do not depend on $\boldsymbol{x}$. Isotropy means that $\xi(r)$ does not depend on the direction $\boldsymbol{n}$.[4] An important application we have in mind are cosmological density fluctuations, $\delta(\boldsymbol{x}) = [\rho(\boldsymbol{x}) - \rho_0]/\rho_0$, where $\rho_0 = \langle\rho\rangle$ is the mean density; but the following arguments are completely general.[5] Here and in what follows we assume that the average density ρ_0 is a well-defined positive quantity. This is not so if the distribution is fractal (Pietronero 1987).

Our goal is to determine the correlation function of local maxima from the correlation function of the underlying density field. Like Kaiser (1984), we simplify the problem by computing the correlations of *regions* above a certain threshold $\nu\sigma$ instead of the correlations of *maxima*. However, these quantities are closely related for values of ν significantly larger than 1. We define the threshold density $\theta_\nu(\boldsymbol{x})$ by

$$\theta_\nu(\boldsymbol{x}) \equiv \theta\,[\delta(\boldsymbol{x}) - \nu\sigma] = \begin{cases} 1 & \text{if } \delta(\boldsymbol{x}) \geq \nu\sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Note the qualitative difference between δ, which is a weighted density field, and θ_ν, which just defines a set, all points having equal weight. We note the following simple facts concerning the threshold density θ_ν, due only to its definition, independently on the correlation properties of $\delta(\boldsymbol{x})$:

$$\begin{aligned}
&\langle\theta_\nu\rangle \equiv P_1(\nu) \leq 1, \quad [\theta_\nu(\boldsymbol{x})]^n = \theta_\nu(\boldsymbol{x}), \\
&\langle\theta_\nu(\boldsymbol{x})\theta_\nu(\boldsymbol{x} + r\boldsymbol{n})\rangle \leq P_1(\nu), \\
&\frac{\langle\theta_\nu(\boldsymbol{x})\theta_\nu(\boldsymbol{x} + r\boldsymbol{n})\rangle}{P_1(\nu)^2} - 1 \equiv \xi_\nu(r) \leq \xi_\nu(0) = \frac{1}{P_1(\nu)} - 1, \\
&\theta_{\nu'}(\boldsymbol{x}) < \theta_\nu(\boldsymbol{x}), \quad P_1(\nu') < P_1(\nu) \text{ for } \nu' > \nu, \\
&\xi_{\nu'}(0) > \xi_\nu(0) \text{ for } \nu' > \nu.
\end{aligned} \quad (2)$$

The difference between θ_ν for different values of ν is called biasing. The enhancement of $\xi_\nu(0)$ for higher thresholds has clearly nothing to do with how "strongly clustered" the peaks are but is entirely due to the fact that the larger ν, the lower the fraction of points above the threshold [i.e., $P_1(\nu') < P_1(\nu)$ for $\nu' > \nu$]. If we consider the trivial case of white Gaussian noise [$\xi(r) = 0$ for $r > 0$], the peaks are just spikes. When a threshold $\nu\sigma$ is considered, the number of spikes decreases and hence $\xi_\nu(0)$ is amplified because they are much more sparse and not because they are "more strongly clustered": we show in the following that also in the case of a correlated field [$\xi(r) \not\equiv 0$ for $r > 0$] the importance of sparseness is crucial in order to explain the amplification of $\xi_\nu(r)$.

In the context of cosmological density fluctuations, if the average density of matter is a well-defined positive constant, the amplitude of $\xi_m(r)$ of matter distribution is very important, since its integral over a given radius is proportional to the over density on this scale,

$$\sigma(R) = 3R^{-3} \int_0^R \xi_m(r) r^2\, dr. \quad (3)$$

[1] Laboratoire de Physique de la Matière Condensée, École Polytechnique, 91128-Palaiseau Cedex, France.

[2] INFM Sezione Roma1, Dipartimento di Fisica, Universitá "La Sapienza," Piazza le Aldo Moro, 2, I-00185 Roma, Italy.

[3] Département de Physique Théorique, Université de Genève, 24 quai Ernest Ansermet, CH-1211 Genève 4, Switzerland.

[4] In other words, we assume $\delta(\boldsymbol{x})$ to be a so-called "stationary normal stochastic process" (Feller 1965).

[5] Clearly, cosmological density fluctuations can never be perfectly Gaussian, since $\rho(\boldsymbol{x}) \geq 0$ and thus $\delta(\boldsymbol{x}) \geq -1$, but for small fluctuations, a Gaussian can be a good approximation. Furthermore, our results remain at least qualitatively correct also in the non-Gaussian case.

The scale R_l, where $\sigma(R_l) \sim 1$, separates large, nonlinear fluctuations from small ones (Gaite, Domínguez, & Pérez-Mercader 1999). It is very important to stress the following point: from the knowledge of the functions $\xi_\nu(r)$ for two different subsets of the density field obtained from two different values ν and ν' of the threshold, it is not possible to predict the amplitude of the fluctuations of the original density field at any scale if we do not know the underlying values of ν, ν', and σ. On the other hand, as we are going to show, the only feature of the original field that can be inferred by the behavior of $\xi_\nu(r)$ is the large-scale behavior of the correlation function $\xi(r)$, in particular the correlation length (if this length is finite in the statistical physics terminology). The correlation length r_c can be defined as (Gaite et al. 1999)

$$r_c^2 = \frac{1}{2}\left|\frac{\nabla^2 P(k)}{P(k)}\right|_{k=0}, \tag{4}$$

where $P(k)$ is the Fourier transform of $\xi(r)$. Note that if r_c is independent of any multiplying constant in $\xi(r)$, then it is not related to its amplitude. This correlation length is that used in statistical physics and field theory (Ma 1984) and gives the length scale beyond which $\xi(r)$ decays rapidly to zero (e.g., exponentially). Roughly, this implies that the fluctuations of the field are organized in structures up to a scale r_c (Gaite et al. 1999). However, in cosmology the correlation length has been defined historically (Peebles 1980) through the amplitude of $\xi(r)$ by looking at the distance r_0 at which it is equal to 1. Provided that a constant positive density ρ_0 of the field exists, r_0 gives the scale beyond which the fluctuations become small with respect to ρ_0 (then it is analogous to the previously defined R_l), and hence it provides also the minimal size of a sample of the field giving a good estimate of the intrinsic ρ_0. The confusion between r_c and r_0 (see also Gaite et al. 1999) is at the basis of the misinterpretation of the concept of bias, as we are going to show.

The joint two-point probability density $\mathcal{P}_2(\delta, \delta'; r)$ depends on the distance r between $\boldsymbol{x}$ and $\boldsymbol{x}'$, where $\delta = \delta(\boldsymbol{x})$ and $\delta' = \delta(\boldsymbol{x}')$. For Gaussian fields, $\mathcal{P}_2$ is entirely determined by the two-point correlation function $\xi(r)$ (Rise 1954; Feller 1965):

$$\mathcal{P}_2(\delta, \delta'; r) = \frac{1}{2\pi\sqrt{\sigma^4 - \xi(r)^2}} \times \exp\left\{-\frac{\sigma^2(\delta^2 + \delta'^2) - 2\xi(r)\delta\delta'}{2[\sigma^4 - \xi^2(r)]}\right\}. \tag{5}$$

By definition

$$\xi(r) \equiv \langle\delta(\boldsymbol{x} + r\boldsymbol{n})\delta(\boldsymbol{x})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\delta\, d\delta'\, \delta\delta' \mathcal{P}_2(\delta, \delta'; r). \tag{6}$$

The probability that both δ and δ' are larger than $\nu\sigma$ is

$$P_2(\nu, r) = \int_{\nu\sigma}^{\infty}\int_{\nu\sigma}^{\infty} \mathcal{P}_2(\delta, \delta', r)\, d\delta\, d\delta' \equiv \langle\theta_\nu(\boldsymbol{x})\theta_\nu(\boldsymbol{x} + r\boldsymbol{n})\rangle. \tag{7}$$

The conditional probability that $\delta(\boldsymbol{y}) \geq \nu\sigma$, given $\delta(\boldsymbol{x}) \geq \nu\sigma$, where $|\boldsymbol{x} - \boldsymbol{y}| = r$, is then just $P_2(\nu, r)/P_1(\nu)$. The two-point correlation function for the stochastic variable $\theta_\nu(\boldsymbol{x})$ introduced above can be expressed in terms of P_1 and P_2 by

$$\xi_\nu(r) = \frac{P_2(\nu, r)}{P_1^2(\nu)} - 1. \tag{8}$$

Defining $\xi_c(r) = \xi(r)/\sigma^2$, we obtain

$$P_1(\nu)^2[\xi_\nu(r) + 1] = \frac{1}{2\pi\sqrt{1 - \xi_c^2}}\int_\nu^\infty\int_\nu^\infty dx\, dx' \times \exp\left\{-\frac{(x^2 + x'^2) - 2\xi_c(r)xx'}{2[1 - \xi_c^2(r)]}\right\}. \tag{9}$$

It is worth noting that the amplitude of $\xi_\nu(r)$ does not give information about how large the fluctuations are with respect to ρ_0, but it rather describes the "fluctuations of the fluctuations," that is the fluctuations of the new variable $\theta_\nu(\boldsymbol{x})$ around its average $P_1(\nu)$. Similar arguments to those introduced for the original field can now be developed to characterize the typical scales of the new set defined by $\theta_\nu(\boldsymbol{x})$. In particular, one can define a correlation length $r_c(\nu)$ using the analog of equation (4) by replacing $\xi(r)$ with $\xi_\nu(r)$. Like r_c, $r_c(\nu)$ does not depend on any multiplicative constant in $\xi_\nu(r)$, i.e., it does not depend on the amplitude of $\xi_\nu(r)$. Moreover, a "homogeneity scale" $r_0(\nu)$ can be defined looking at the scale at which $\xi_\nu(r) = 1$ (or alternatively eq. [3]). The value of $r_0(\nu)$ strongly depends on the amplitude of $\xi_\nu(r)$ and represents the minimal size of a sample of the set giving meaningful estimates of the average density $P_1(\nu)$ and of $r_0(\nu)$ itself; $r_0(\nu)$ is the distance at which the conditional density $P_2(\nu, r)/P_1(\nu)$ begins to flatten toward $P_1(\nu)$. We show below that while $r_0(\nu)$ depends strongly on ν because of a sparseness effect, $r_c(\nu)$ is almost constant and equal to r_c of the field, i.e., the maximal size of the fluctuations' structures does depend on the threshold.

Equation (9) implies, for $\nu \gg 1$ and for sufficiently large r such that $\xi_c(r) \ll 1$ (Politzer & Wise 1984),

$$\xi_\nu(r) \simeq \exp[\nu^2\xi_c(r)] - 1 \tag{10}$$

to lowest nonvanishing order in $\xi_c(r)$. If, in addition, $\nu^2\xi_c(r) \ll 1$, we find (Politzer & Wise 1984)

$$\xi_\nu(r) \simeq \nu^2\xi_c(r). \tag{11}$$

This is the relation derived by Kaiser (1984). He only states the condition $\xi_c(r) \ll 1$ and separately $\nu \gg 1$, which is significantly weaker than the required $\nu^2\xi_c(r) \simeq \xi_\nu(r) \ll 1$, especially around the correlation length where ξ is not yet very small.

It is important to note that in the cosmologically relevant regime, $\xi_\nu \gtrsim 1$, the Kaiser relation (eq. [11]) does not apply and ξ_ν is actually exponentially enhanced. If this mechanism would be the cause for the observed cluster correlation function, one would thus expect an exponential enhancement on scales

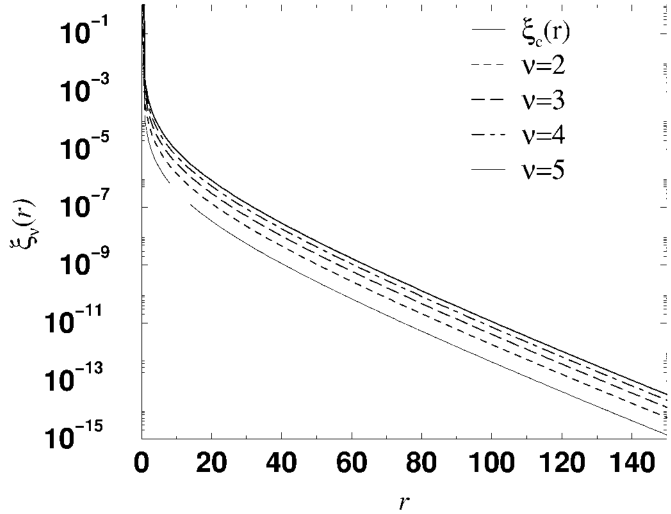

FIG. 1.—Behavior of $\xi(r) = \sigma^2/[1 + (k_s r)^\gamma \exp(-r/r_c)]$ (where $\gamma = -2$, $k_s^{-1} = 0.01$, and $r_c = 10$) and $\xi_\nu(r)$ are shown for different values of the threshold ν in a semilog plot. The slope of $\xi_\nu(r)$ for $r \gtrsim 50$ is $-1/r_c$, independent of ν, i.e., the correlation length of the system does not change for the sets above the threshold.

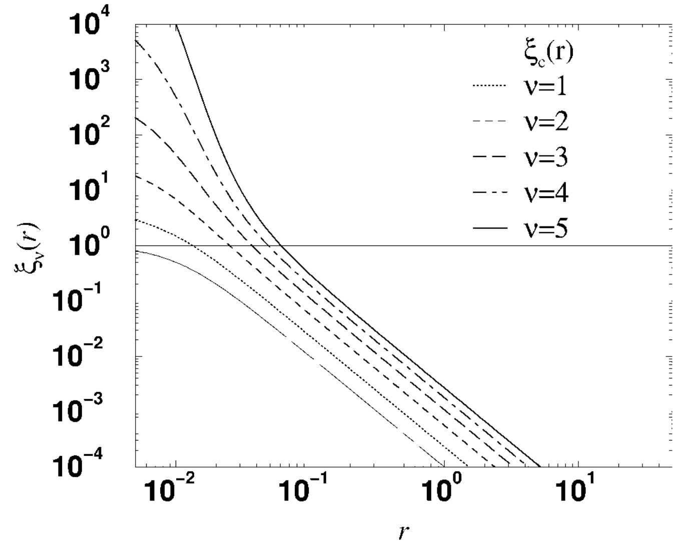

FIG. 2.—Behavior of $\xi(r) \sim \sigma^2/[1 + (k_s r)^\gamma]$ (with $\gamma = -2$, $k_s^{-1} = 0.01$) and $\xi_\nu(r)$ are shown for different values of the threshold ν in a log-log plot.

at which $\xi_{cc} \gtrsim 1$, i.e., $R \lesssim 20\ h^{-1}$ Mpc. This is in contradiction with observations (Bahcall & Soneira 1983)![6]

If, within a range of scales, $\xi(r)$ can be approximated by a power law $\xi = (r/r_0)^{-\gamma}$, and if the threshold ν is such that equation (11) holds, which implies $\xi_\nu \ll 1$, we have $\xi_\nu = [r/r_0(\nu)]^{-\gamma}$. The scales $r_0(\nu)$ for different biases are related by $r_0(\nu') = r_0(\nu)(\nu'/\nu)^{2/\gamma}$. For that reason Kaiser, who first derived equation (11), interpreted it as an increase in the "correlation length" $r_0(\nu)$, which in our language is the homogeneity scale of the set $\theta_\nu(\boldsymbol{x})$.

In order to clarify the meaning of the two length scales $r_c(\nu)$ and $r_0(\nu)$, we first study an example of a Gaussian density field with finite correlation length r_c and which is well approximated by a power law on a certain range of scales. The case in which $r_c \to \infty$ is straightforward. We set

$$\xi(r) = \frac{\sigma^2 \exp(-r/r_c)}{1 + (k_s r)^\gamma}$$

with $k_s^{-1} \ll r_c$; k_s^{-1} represents the smoothing scale of the continuous field, which is characterized better in the following, and r_c is approximately the correlation length as defined as equation (4). In the region $k_s^{-1} \ll r \ll r_c$, $\xi(r)$ is well approximated by the power law $(k_s r)^{-\gamma}$. The correlation lengths, $r_c(\nu)$ for any value of ν, are given by the slope of $\log \xi_\nu(r)$ at large r versus r, which is clearly independent of bias (Fig. 1). This can also be obtained from equations (10) and (11).

For relatively small values of the threshold, $\nu \ll \nu_c \approx (k_s r_c)^{\gamma/2}$, one finds in this case $r_0(\nu) \ll r_c$ and $r_0(\nu) \sim k_s^{-1}\nu$. On the other hand, if $\nu \gg \nu_c$, we have $r_0(\nu) \sim r_c \log(\nu)$, and in this case the statistics are dominated by shot noise (see below). For this reason we assume $r_0(\nu) < r_c(\nu)$ in the following. We note that in the range of scales $r \leq r_0(\nu)$, the amplification of $\xi_\nu(r)$ is strongly nonlinear in ν and is scale dependent: hence if the original correlation function $\xi(r)$ has a power-law behavior, $\xi_\nu(r)$ does not for $r \leq r_0(\nu)$. This is better shown in the case in which $r_c \to \infty$. In this case, the correlation function is

$$\xi(r) = \frac{\sigma^2}{1 + (r k_s^{-1})^\gamma}. \quad (12)$$

Clearly on scales $k_s^{-1} < r < r_c$ this example does not differ from the above (but of course the correlation length is infinite here). The amplification of ξ_ν for this example is plotted in Figure 2. In order to investigate whether $\xi_\nu(r)$ is of the form $\xi_\nu(r) \sim [r/r_0(\nu)]^{-\gamma_\nu}$, we plot $-d \log[\xi_\nu(r)]/d \log(r) \sim \gamma_\nu$ in Figure 3. Only in the regime in which $\xi_\nu(r) \ll 1$ does γ_ν become constant

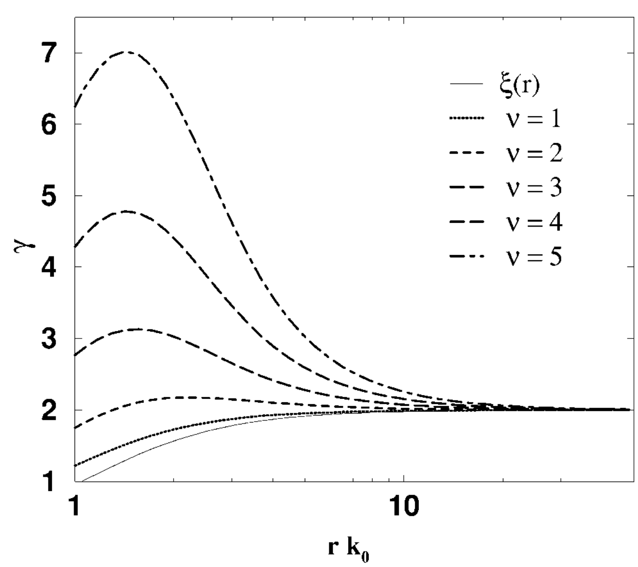

FIG. 3.—Behavior of $\gamma_\nu(r)$ is shown for different values of the threshold ν for the correlation function shown in Fig. 2. Clearly γ_ν is strongly scale dependent on all scales on which $\xi_\nu \gtrsim 1$ (this is $r < 1$ in our units).

[6] One might argue that nonlinearities which are important when the fluctuations are large can "rescue" the Kaiser relation (eq. [11]) also into the regime $\xi_\nu > 1$. There are two objections against this: First of all, as we pointed out above, $\xi_\nu > 1$ does not imply large fluctuations of the original density field. Actually, most cosmologists would agree that on $R \sim 20\ h^{-1}$ Mpc, where the cluster correlation function $\xi_{cc} \sim 1$, fluctuations are linear. Second, it seems very unphysical that Newtonian clustering should act as to change the exponential relation (eq. [10]) into a linear one (eq. [11]).

and roughly independent of ν. This behavior is very different from the one found in galaxy catalogs!

Let us now clarify how the amplification of $\xi_\nu(r)$ is related to the increase of the peak sparseness with the threshold ν. For a Gaussian random field, the mean peak size $D_p(\nu)$ and the mean peak distance L_p are, respectively (Vanmarcke 1983; Coles 1986), $D_p(\nu) \simeq D_0(k_s, r_c)/\nu$ and $L_p(\nu) \simeq D_0(k_s, r_c) \times \exp(\nu^2/6)\nu^{-2/3}$ so that

$$L_p/D_p \simeq \nu^{1/3} \exp(\nu^2/6) \text{ for } \nu \gg 1. \quad (13)$$

$D_0(k_s, r_c)$ is given by

$$D_0^2 = \frac{\int_0^{+\infty} dk\, P_1(k)}{\int_0^{+\infty} dk\, k^2 P_1(k)}, \quad (14)$$

where $P_1(k)$ is the Fourier transform of $\xi(r)$ along a line in space [in $d = 1$, it coincides with $P(k)$]. Equation (13) shows the strong enhancement of the sparseness of peaks (object) with increasing ν. It is this increase of sparseness that is at the origin of the amplification by biasing. In light of equations (10), (11), and (13), we see that increasing ν corresponds to a very particular sampling of fluctuations: the typical size of the surviving peaks D_p is slowly varying with ν, while the average distance between peaks L_p is more than exponentially amplified, and finally the scale $r_c(\nu)$, over which the fluctuations are structured, is practically unchanged.

We have argued that bias does not influence the correlation length [$r_c(\nu) \simeq r_c$]. It amplifies the correlation function by the fact that the mean density $P_1(\nu)$ is reduced more strongly than the conditional density $P_2(\nu, r)/P_1(\nu)$. According to equation (10), this amplification is strongly nonlinear in $\xi(r)$ (exponential) at scales at which $\nu^2\xi_c(r) \geq 1$ and thus $\xi_\nu(r) > 1$.

Consequently, as we want to stress once more, the biasing mechanism introduced by Kaiser and discussed in this work cannot lead to a relation of the form $\xi_{\nu'}(r) = \alpha_{\nu'\nu}\xi_\nu(r)$ over a range of scales $r_1 < r < r_2$ such that $1 < \xi_\nu(r_1)$ and $\xi_\nu(r_2) < 1$. But exactly this behavior is found in galaxy and cluster catalogs. For example, in Bahcall & Soneira (1983) or Benoist et al. (1996), a constant biasing factor $\alpha_{\nu'\nu}$ over a range from about 1 to 20 h^{-1} Mpc is observed for correlation amplitudes varying from about 20 to 0.1. We therefore conclude that the explanation by Kaiser (1984) cannot be at the origin of the difference of the correlation functions observed in the distribution of galaxies with different intrinsic magnitude or in the distribution of clusters with different richness.

This result appears at first disappointing, since it invalidates an explanation without proposing a new one. On the other hand, the search for an explanation of an observed phenomenon is only motivated if we are fully aware of the fact that we do not already have one.

Last but not least, we want to point out that fractal density fluctuations together with the fact that more luminous objects are seen out to larger distances do actually induce an increase in the amplitude of the correlation function $\xi(r)$ similar to the one observed in real galaxy catalogs (Pietronero 1987; Sylos Labini, Montuori, & Pietronero 1998). In this explanation, the linear amplification found for the correlation function has nothing to do with a correlation length but is a pure finite size effect, and the distribution of galaxies does not have any intrinsic characteristic scale.

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