Laplacian Fractal Growth in Media with Quenched Disorder

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We study the combined effect of a Laplacian field and quenched disorder in the generation of fractal structures in the quenched dielectric breakdown model. The growth dynamics is shown to evolve from the avalanches of invasion percolation (IP) to the smooth growth of Laplacian fractals, i.e., diffusion limited aggregation and the dielectric breakdown model (DBM). The fractal dimension is strongly reduced with respect to both the DBM and IP, due to the combined effect of memory and field screening. This implies a specific relation between the fractal dimension of the breakdown structures and the microscopic properties of disordered materials. [S0031-9007(97)03890-8]

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The growth of fractal structures is usually described in terms of physical models characterized by an irreversible dynamics and a degree of self-organization. These models can be divided in two broad classes. In the first one the dynamics is *stochastic* (modulated by a field), and its prominent examples are diffusion limited aggregation (DLA) [1], the dielectric breakdown model (DBM) [2], and their variants. The second class is characterized by a deterministic dynamics in a *quenched* random medium. An example is invasion percolation (IP) [3] and a related model is the Bak and Sneppen (BS) [4] model of biological evolution. Concerning the growth dynamics, some of these models grow in a smooth way (i.e., DLA and DBM), while other ones grow by *avalanches* (i.e., IP) [5].

Our understanding of the origin of fractal structures for the two classes is based on the idea that an *effective screening* is present in the scale invariant dynamics [6]. However, the origin and the properties of this screening are very different. For Laplacian fractals it arises from the geometrical screening of the electric field around the structure. For the case of quenched dynamics, instead, there is no field and an effective screening develops as a memory effect from the quenched dynamics itself.

The two kinds of dynamical models, stochastic with a field and quenched without a field, represent, however, two extreme limiting cases of real natural phenomena which usually present both features. For example, dielectric breakdown and fracture propagation in disordered solids represent important cases in which these elements are combined [7].

In this Letter we address the question of understanding how the combination of these two basic elements operates and leads to new phenomena. From a theoretical point of view this requires the unification of the concepts developed for the two limiting cases. Together with suitable simulations, this allows us to link the microscopic properties of disordered materials to the resulting fractal structures.

In the original DBM the local field E_i across a perimeter bond i of the structure is related to the probability of growth by $p_i \propto E_i^\eta$, where η is a parameter characterizing the strength of the link between the field and the growth probability. The growth probability of a bond is then normalized with that of all the other perimeter bonds. If we consider the generalization of such a process in a random medium [7] each bond can be associated with a quenched variable $x_i \in [0,1]$, extracted from a given probability distribution, which defines the strength of the bond (resistivity) with respect to the breakdown. The growth process proceeds by breaking the weakest bond. Without the field this means to break the bond with $\min_i\{x_i\}$; this is invasion percolation [3]. If we introduce the local field E_i, the tendency to break will be modulated by the field itself. This requires the introduction of a new variable,

$$y_i(t) = \frac{x_i}{[E_i(t)]^\eta}, \tag{1}$$

which takes into account this effect. Growth at time t occurs at the bond with the smallest $y_i(t)$.

The probability density of x_i, following Ref. [7], is

$$p_0(x) = ax^{a-1}, \tag{2}$$

where the parameter a ($a \in [0,\infty]$) models the strength of the material. Small values of a ($a \ll 1$) correspond to a fragile material in which most bonds are easy to break. Large values of a ($a \gg 1$) correspond instead to a strong material. The limit $a \to \infty$ ($\eta \to \infty$) implies that growth occurs deterministically at the bond with the largest field. The case $\eta = 0$ eliminates the effect of the field and leads to the IP model. On the other hand, there is no obvious limit that brings us back to the original (stochastic) DBM. In this respect the present model that we may call quenched DBM (QDBM) contains the elements of both the DBM and IP models, but it has a simple limiting case only towards IP.

The effect of the field on the quenched variables is to modify the extension of its distribution for a given bond, as shown in Fig. 1. So, while the extension of x_i is on the $[0,1]$ interval for all bonds, the distribution of y_i will have an extension depending on the local field E_i (Fig. 1). It is convenient to use a normalized field (at time t)

$$[E_i'(t)]^\eta = \left[\frac{E_i(t)}{E_{\max}(t)}\right]^\eta, \quad (3)$$

where $E_{\max}(t)$ is the maximum value of the field among the perimeter bonds at time t. Using this field we have $y_i(t) = x_i/[E_i'(t)]^\eta$, with a probability density [from Eq. (2)]

$$p_0(y) = a[E_i'(t)]^{a\eta} y^{a-1}, \quad (4)$$

and therefore the range of the variable $y_i(t)$ for the bond with the largest field is always $0 \leq y_i(t) \leq 1$, which is more convenient for the analysis.

A fundamental characterization of the growth process comes from the acceptation profiles, corresponding to the distributions of the x_i, y_i for the grown bonds. In Fig. 2 we report, for parameter values $a = 1$ and $\eta = 1$, the distribution of the variables x_i of the grown bonds, while the inset refers to the analogous distribution for the y_i.

The behavior of these distributions as a function of the total growth time t shows the *self-organization* of the system towards specific limiting distributions. The distribution of the x_i converges to a function which extends on the entire $[0,1]$ interval. This implies that a value of x_i close to 1 (unfavorable) may actually prevail on all others because of the effect of the field. This situation is very different from the case of IP, in which the analogous distribution is a theta function with a discontinuity at $x = p_c < 1$, where p_c is the critical bond percolation probability. The existence of such a threshold implies that the dynamics evolves by scale invariant avalanches [5,8].

The acceptation profile of the y_i (inset of Fig. 2) seems to converge to a theta function, but its meaning is quite different from the case of IP. In fact, the distribution must be zero for $y > 1$ by construction, because at least the variable with the maximum field must be smaller than 1. Therefore the bonds that are allowed to grow must refer to values of y in $[0,1]$. On the other hand, its approximately flat behavior is quite nontrivial, and it indicates that the effect of the field dominates with respect to that of the quenched disorder. The best candidates for growth are near the tips because of the field, so that memory effects are limited with respect to IP and the process is close to deterministic DBM (which would give $D = 1$).

Sample structures of QDBM growth are shown in Fig. 3. They were found to satisfy fractal scaling with a definite fractal dimension which is reported in Table I for different values of a and η. These give a strong numerical evidence for the invariance property

$$D_f(\eta; a) = D_f(\eta a). \quad (5)$$

This result is quite interesting and nontrivial in view of the very different role that the parameters η and a play in the growth process. Later we are going to see theoretically what is the origin of this symmetry property. Moreover, we found that these values of fractal dimensions do not depend on the sizes of the simulation. In fact, for $a = 1$ and $\eta = 0.2$ fixed, and different extensions of the base $L = 128$, $L = 256$, and $L = 512$, we found, respectively, $D_f = 1.3 \pm 0.1$, $D_f = 1.34 \pm 0.04$, and $D_f = 1.33 \pm 0.02$.

From the theoretical side the growth process is deterministic, and it arises from the quenched disorder modulated by the field of the structure itself. A suitable approach to deal with a problem of fractal growth with quenched disorder is the method of the run time statistics (RTS) [8–10]. It consists in the mapping of a quenched dynamics into a stochastic one with *cognitive memory*. The basic concept of the mapping is the following: if a bond has "lost" many times, there is a finite probability

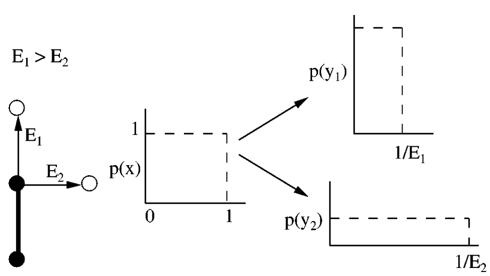

FIG. 1. Schematic picture of the role of a field (E_i) on the dynamics of QDBM. The two bonds have the same (flat) distribution for the variable x_i, which represents the microscopic properties of the material. However, once the bonds are subjected to different fields ($E_1 > E_2$), the effective variables for the breakdown process become the $y_i = x_i/E_i$, whose distributions are modulated by the field.

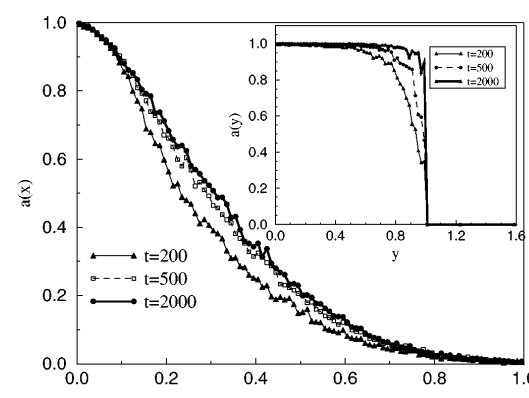

FIG. 2. Time evolution of the acceptation profiles for the variables x_i and y_i (inset) for the bonds which grow. For $t \rightarrow \infty$ the convergence towards limiting distributions corresponds to the self- organized nature of the dynamics. The absence of a threshold value for $a(x)$ implies a smooth growth (not by avalanches).

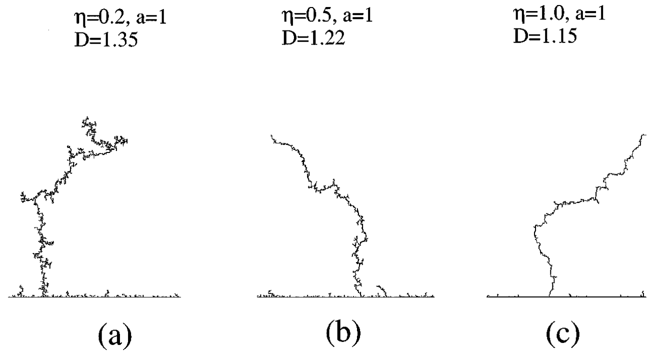

FIG. 3. Examples of QDBM clusters for different values of $a \cdot \eta$. Note the small values of D with respect to IP and DBM.

that it will never grow, even after an infinite time. This introduces an effective dynamical screening which is at the basis of fractal properties. The RTS method provides a systematic technique to describe this phenomenon, which is typical of models with quenched disorder.

In the present case the situation is more complex in view of the modulation induced by the field. To this purpose it is convenient to introduce the following concepts. To each interface bond we assign an effective probability density $p_{t,i}(y)$, where i gives the position of the bond and t is the total growth time. As soon as a bond becomes part of the growing interface, its distribution is the original one $p_{t,i}(y) = p_0(y) = a[E_i'(t)]^{a\eta}y^{a-1}$. After a time τ the density is modified in a way that is conditional to the growth history of the bond. τ here can be considered as the "age" of the variable and, as a result of extreme dynamics, the distribution of the variable x_i acquires a dependence on its age. This dependence can be quantified using the RTS method. Following [8], we can derive an expression for the probability that y_i is the extremal variable, which is the growth probability $\mu_{t,i}$ of the variable i at time t [11]

$$\mu_{t,i} = \int_0^1 dy\, p_{t,i}(y) \prod_{m\neq i} \int_y^{a_m(t)} dy_m\, p_{t,m}(y_m), \qquad (6)$$

where $a_m(t) = [E_m'(t)]^{-\eta}$ and the product accounts for the competition between the growing bond and the other perimeter bonds. In the same way we can obtain an equation for the update from time t to $t+1$ of the effective density of bond j after site i has grown,

$$p_{t+1,j}(y) = \left[r_j p_{t,j}(r_j y) \int_0^{r_j y} dy_i\, p_{t,i}(y_i)\theta\big(a_i(t) - y_i\big) \prod_{m\neq j,i} \theta\big(a_m(t) - y_i\big) \int_{y_i}^{a_m(t)} dy_m\, p_{t,m}(y_m) \right] \frac{1}{\mu_{t,i}}, \qquad (7)$$

where $r_j = [E_j'(t+1)/E_j'(t)]^\eta$ accounts for the change of the electric field of the variable j from time t to time $t+1$.

Equations (6) and (7) provide the mapping of the original quenched problem into a stochastic one defined by growth probabilities. The specific realizations of this stochastic process have the same statistical weight as those of the original quenched problem. The approximation in this mapping consists in neglecting the geometrical correlations between probability densities of different bonds, but this can be shown to be exact in the limit of large systems [12]. Other approximations of technical nature will be necessary for the practical use of this mapping in the fixed scale transformation (FST) scheme for the calculation of the fractal dimension.

From the expression (6) it is possible to derive the invariance property (5) that we inferred from the simulations. In fact, one can show by induction that $\mu_{t,i}(a;\eta) = \mu_{t,i}(a\eta)$ [11]. We can also study the behavior of screening effects in QDBM dynamics. An analysis of Eqs. (6) and (7) gives [11]

$$\mu_{t,i} \propto [E_i'(t)]^{a\eta(t-t_0)}, \qquad (8)$$

where t_0 is the time where the bond first became part of the interface. The electric field is characterized by an exponential screening which, for growth models [13], yields $E_i'(t) \propto e^{-c(t-t_0)}$. Therefore, in terms of the age $\tau_i = t - t_0$, Eq. (8) becomes

$$\mu_{t,i} \propto e^{-ca\eta\tau_i^2}. \qquad (9)$$

This result shows that screening effects in QDBM are *stronger than in stochastic DBM* (where $\mu_{t,i} \propto e^{-c\tau_i}$ [13]). This explains therefore why QDBM has a fractal dimension much smaller than the usual DBM or DLA. Moreover, the exponential screening in QDBM, compared to the power law screening $[\mu_{t,i} \propto (\tau_i + 1)^{-\alpha}$ [10]] corresponding to IP and similar models, gives a further evidence for the absence of scale invariant avalanches in IP. Indeed, as discussed in [8], the form of the screening in extremal processes determines the shape of the corresponding avalanche size distribution. Therefore, a power law distribution for the avalanche sizes can only arise from a power law screening of the growth probability of the interface bonds with respect to the growth time.

Having characterized the quenched dynamics of the QDBM in terms of RTS growth probabilities, we can now consider the use of the FST [14] method to analyze the fractal properties of the resulting structures. Such a

TABLE I. Fractal dimension of QDBM clusters of size 512×2048, in cylinder geometry, for different values of the parameters a and η.

η	$D(\eta; a = 1)$	a	$D(a; \eta = 1)$
0.2	1.33 ± 0.02	0.2	1.35 ± 0.02
0.5	1.21 ± 0.02	0.5	1.22 ± 0.02
1.0	1.15 ± 0.02	1.0	1.15 ± 0.02
2.0	1.07 ± 0.01	2.0	1.06 ± 0.01
3.0	1.02 ± 0.01	3.0	1.03 ± 0.01

TABLE II. Second order FST computation of the fractal dimension of QDBM, for different values of η with $a = 1$.

η	$D(\eta)$
0.2	1.70
0.5	1.56
1.0	1.37
2.0	1.23
3.0	1.15

step, however, is nontrivial, and it requires the following considerations. The original dynamics is quenched as in IP, but in the end it does not lead to a critical threshold for the effective probability distribution. This implies that we cannot consider the growth by avalanches as in IP, which would correspond to a specific implementation of the FST [8]. On the contrary, the Gaussian screening of Eq. (9) implies that a fast convergence of the FST matrix elements will be achieved using the same scheme as in DLA and DBM [14].

A nontrivial problem, instead, is the identification of the *scale invariant dynamics* for this problem. In IP the simple extremal rules and the avalanche dynamics can be used to argue that the small scale dynamics is already scale invariant [8]. In DLA (DBM) the situation is more complex, and it requires a specific renormalization study which also showed, however, that the small scale dynamics is rather close to the scale invariant one [6]. In the present case a systematic study of the scale invariant dynamics would require the combined renormalization of the effective probability densities and of the electric fields of the bonds. At the moment, this appears far too complex. So, inspired by the results of the two cases we can treat, we assume that the small scale dynamics is a reasonable approximation to the scale invariant one.

In this way we can proceed to the explicit calculation of the FST matrix elements and of the corresponding fractal dimensions. The results are reported in Table II for different values of η with $a = 1$ (the calculation is up to the second order in the FST scheme).

These results allow us to understand the strong reduction of the fractal dimension for QDBM ($D = 1.37$ for $\eta = 1$) with respect to both IP ($D = 1.8879$ [8]) and DBM ($D = 1.6406$ for $\eta = 1$ and cylinder geometry [14]). On the other hand, the values of D we obtain from theory appear to be somewhat larger that those of the simulations. One can speculate that this may be due to the approximation used for the scale invariant dynamics.

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