The Dynamics of Consensus and Clash

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Consensus: the voter model

Clash:

Bounded compromise: *interaction only if sufficient compatibility*

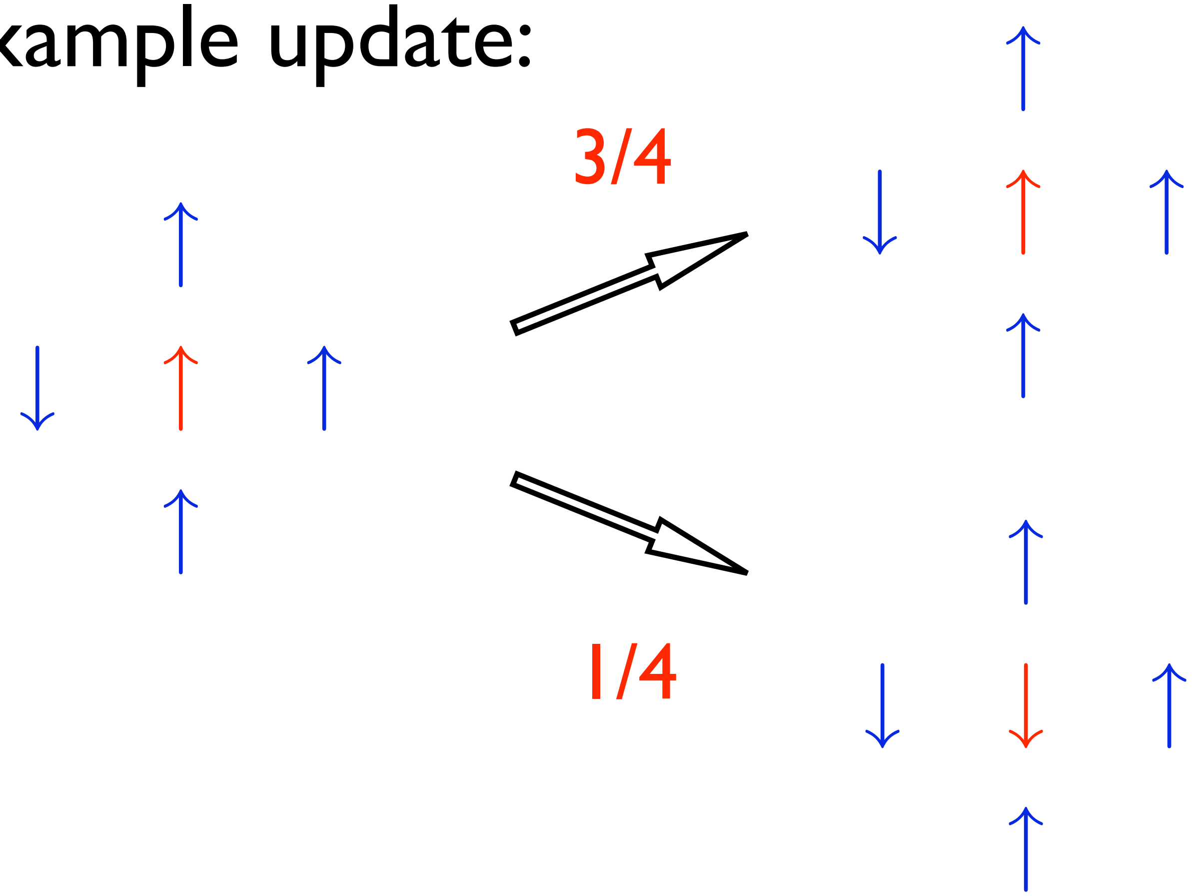
→ *self-similar bifurcation/fragmentation*

Dynamics of social balance: *evolution of social networks with friendly & unfriendly relationships*

→ *dynamical transition between bipolarity and utopia*

Voter Model Liggett (1985)

Example update:



encoded by flip rate

$$w\left(\{\sigma\}_i \rightarrow \{\sigma\}_i'\right) = \frac{1}{2}\left(1 - \frac{\sigma_i}{z}\sum_{k\,\mathrm{nn}\,i} \sigma_k\right)$$

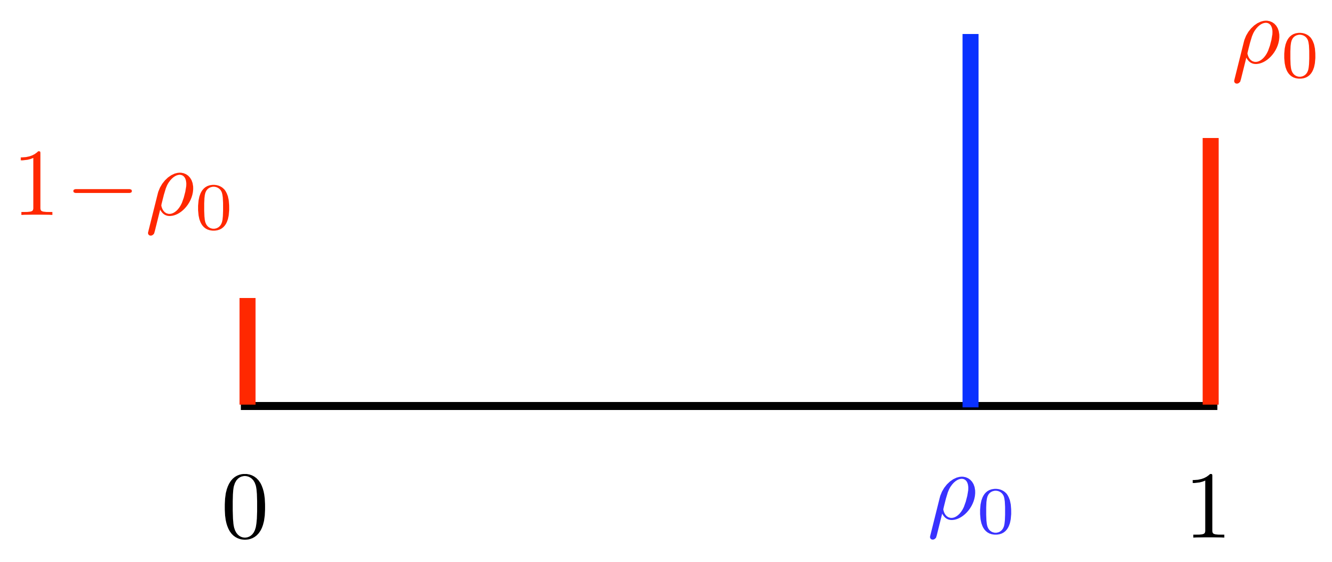
0. Binary voter variable at each site i, $\sigma_i = \pm 1$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   each individual has zero self-confidence and adopts state of randomly-chosen neighbor
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

1. Final State (Exit) Probabilities $E_{\pm}(\rho_0) = \rho_0$

Equation of Motion for single spin:

mean spin $s_i = \langle \sigma_i \rangle$: $\frac{ds_i}{dt} = -2\langle \sigma_i w_i \rangle = -s_i + \frac{1}{z}\sum_{k \text{ nn } i} s_k$

$$\rightarrow \dot{m} \equiv \sum_i \dot{s}_i = 0$$



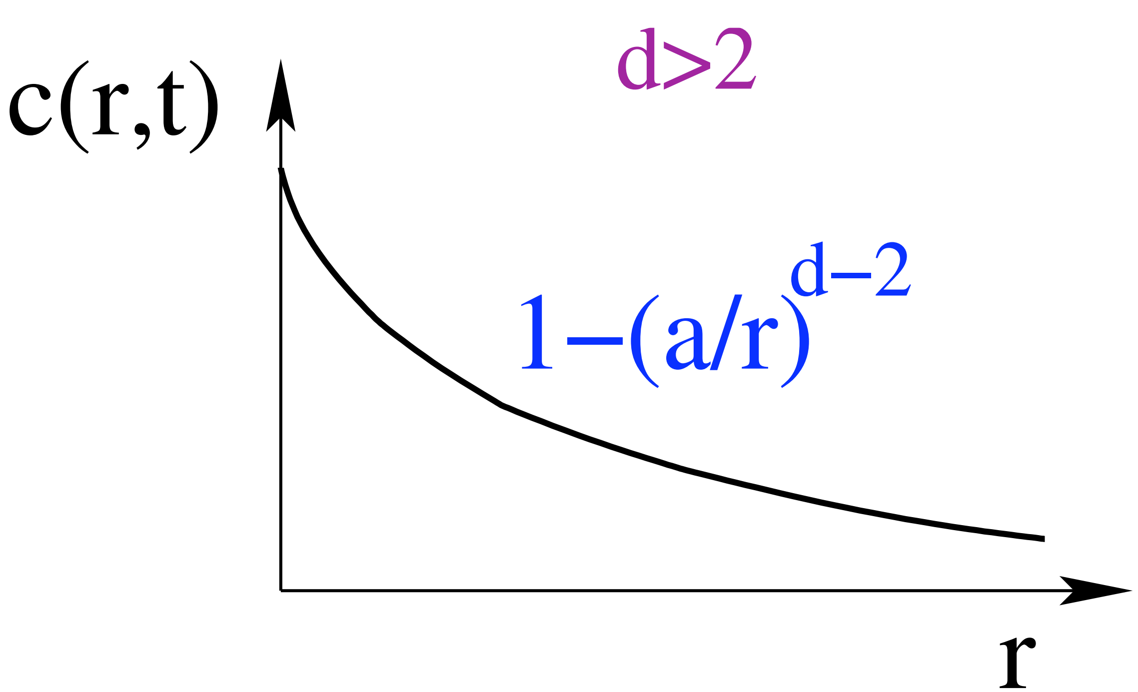
2. Spatial Dependence of 2-Spin Correlations

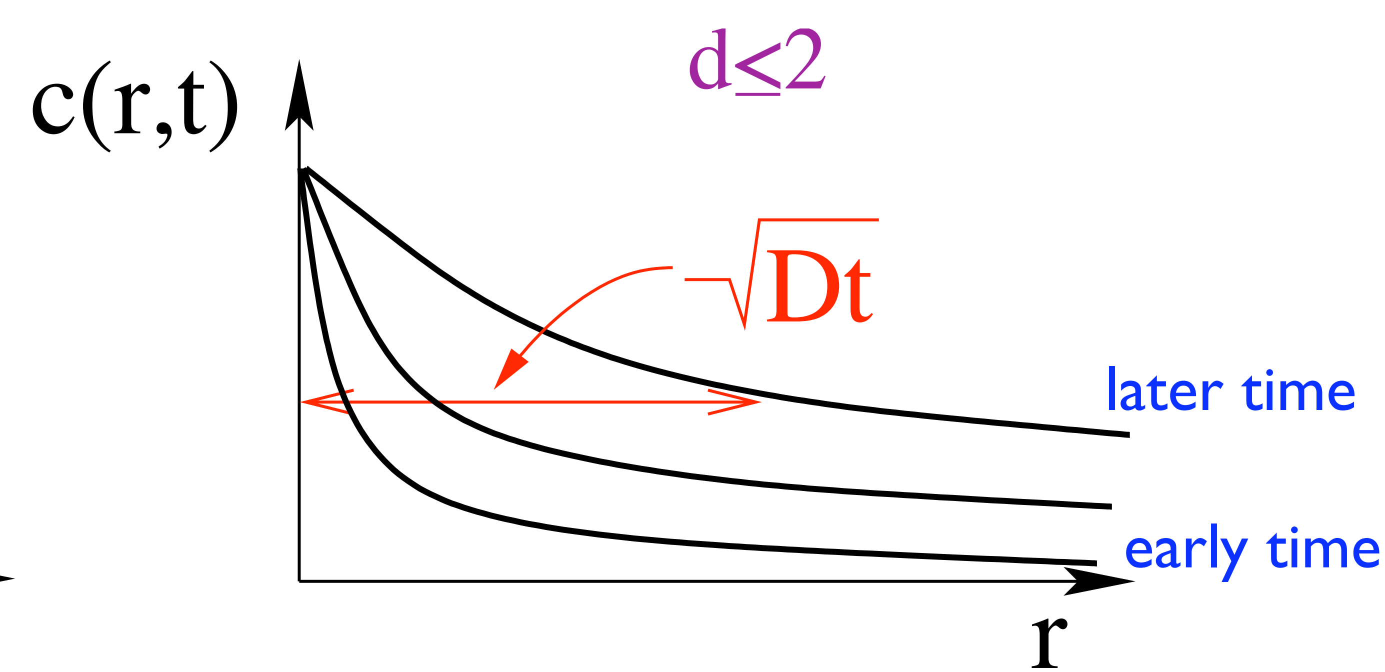
(infinite system)

Equation for 2-spin correlation function:

$$\frac{d(s_i s_j)}{dt} = -2\langle \sigma_i \sigma_j (w_i + w_j) \rangle \rightarrow \frac{\partial c_2(r,t)}{\partial t} = \nabla^2 c_2(r,t)$$

$$c(r=0,t) = 1; \quad c(r>0, t=0) = 0$$





3. System Size Dependence of Consensus Time

Liggett (1985), Krapivsky (1992)

$$\int^{\sqrt{Dt}} c(r,t) r^{d-1}\, dr = N$$

dimension	consensus time
1	N^2
2	N ln N
>2	N

Voter Model on Heterogeneous Networks

V. Sood & SR, PRL **72**, 178701 (2005)

$$n_j = \text{fraction of nodes of degree } j$$

$$\mu_m = \sum_j j^m\, n_j = m^{\text{th}} \text{ moment of degree distribution}$$

$$\omega = \frac{1}{\mu_1}\sum_j j n_j \rho_j = \text{degree-weighted up spin density}$$

Basic result: $T_N(\omega) = -N\frac{\mu_1^2}{\mu_2}\left[(1-\omega)\ln(1-\omega) + \omega\ln\omega\right]$

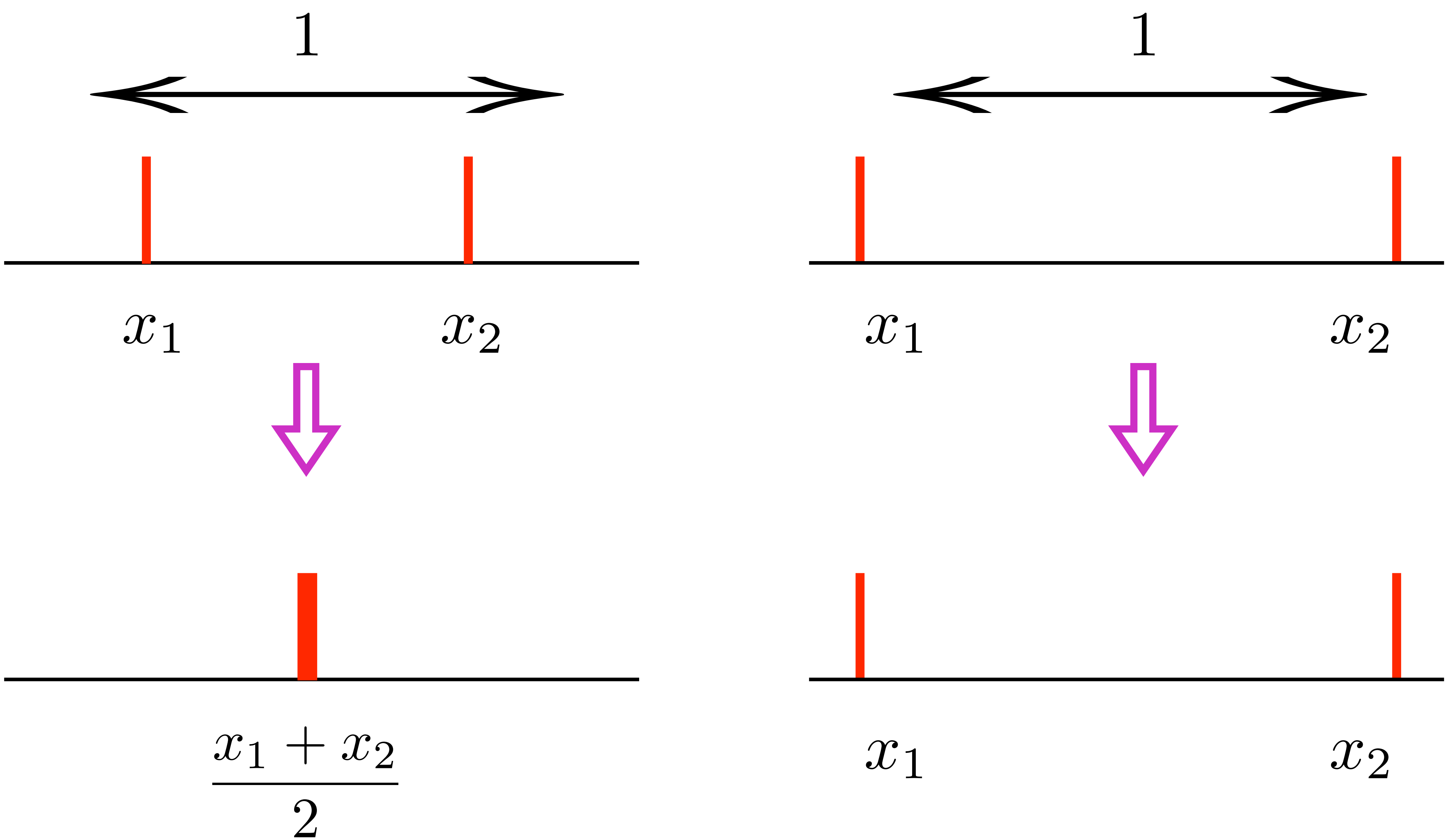
For power-law network: $(n_j \sim j^{-\nu})$

$$T_N \sim \begin{cases} N & \nu > 3, \\ N/\ln N & \nu = 3, \\ N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\ (\ln N)^2 & \nu = 2, \\ \mathcal{O}(1) & \nu < 2. \end{cases}$$

quick consensus!

Bounded Compromise Model

Weisbuch et al (2001)



If $|x_2 - x_1| < 1$ compromise

If $|x_2 - x_1| > 1$ no interaction

The Opinion Distribution

E. Ben-Naim, et al., Physica D **183**, 190 (2003)

Basic observable: $P(x,t) \equiv$ probability that agent has opinion x

Fundamental parameter: Δ, the initial opinion range

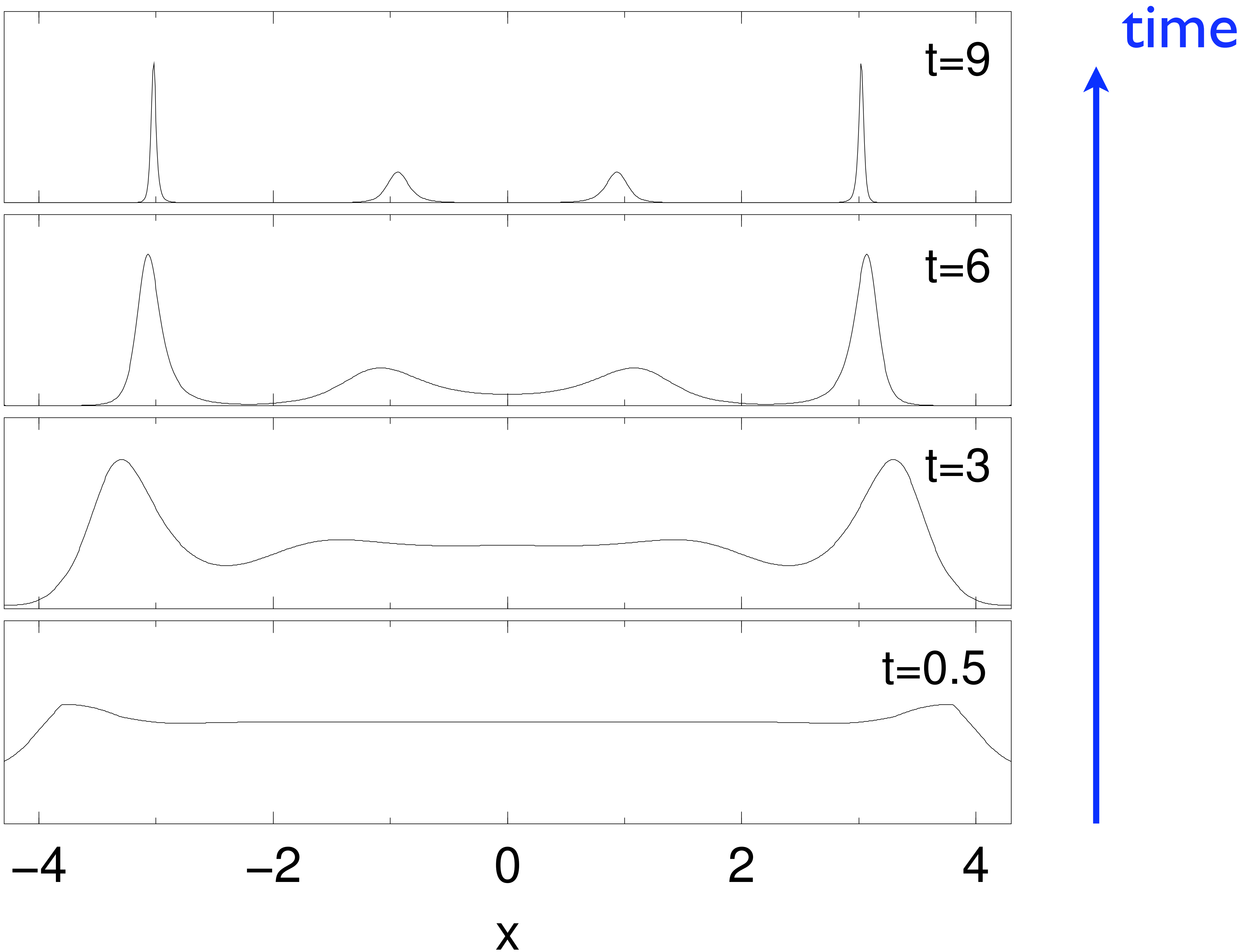
$\Delta<1$: eventual consensus

$\Delta>1$: disjoint "parties"

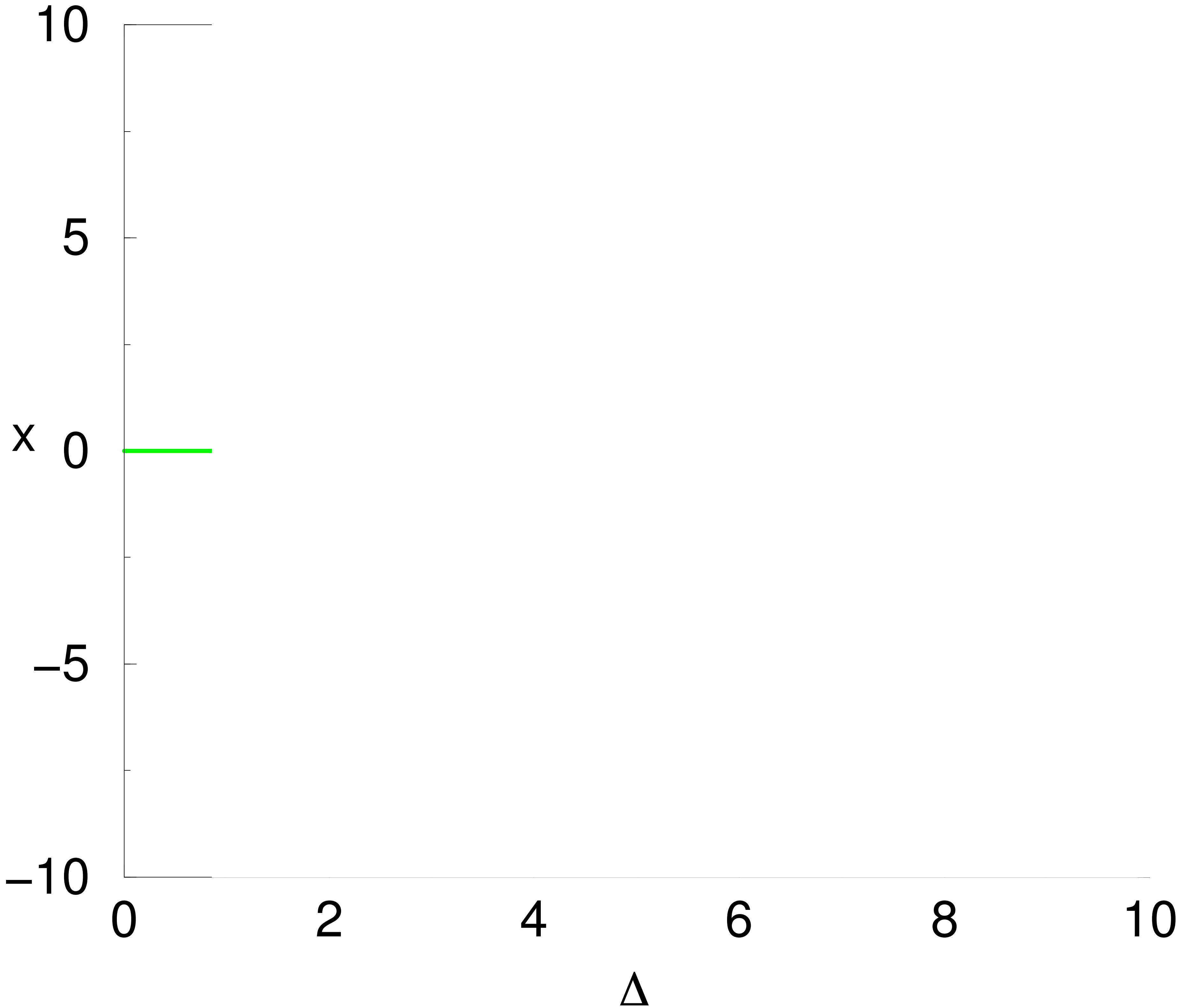
Master equation:

$$\frac{\partial P(x,t)}{\partial t} = \int\int_{|x_1-x_2|<1} dx_1\, dx_2\, P(x_1,t)P(x_2,t) \times \left[\delta\left(x-\frac{x_1+x_2}{2}\right)-\delta(x-x_1)\right]$$

Early time evolution (for Δ=4.3)



Bifurcation Sequence



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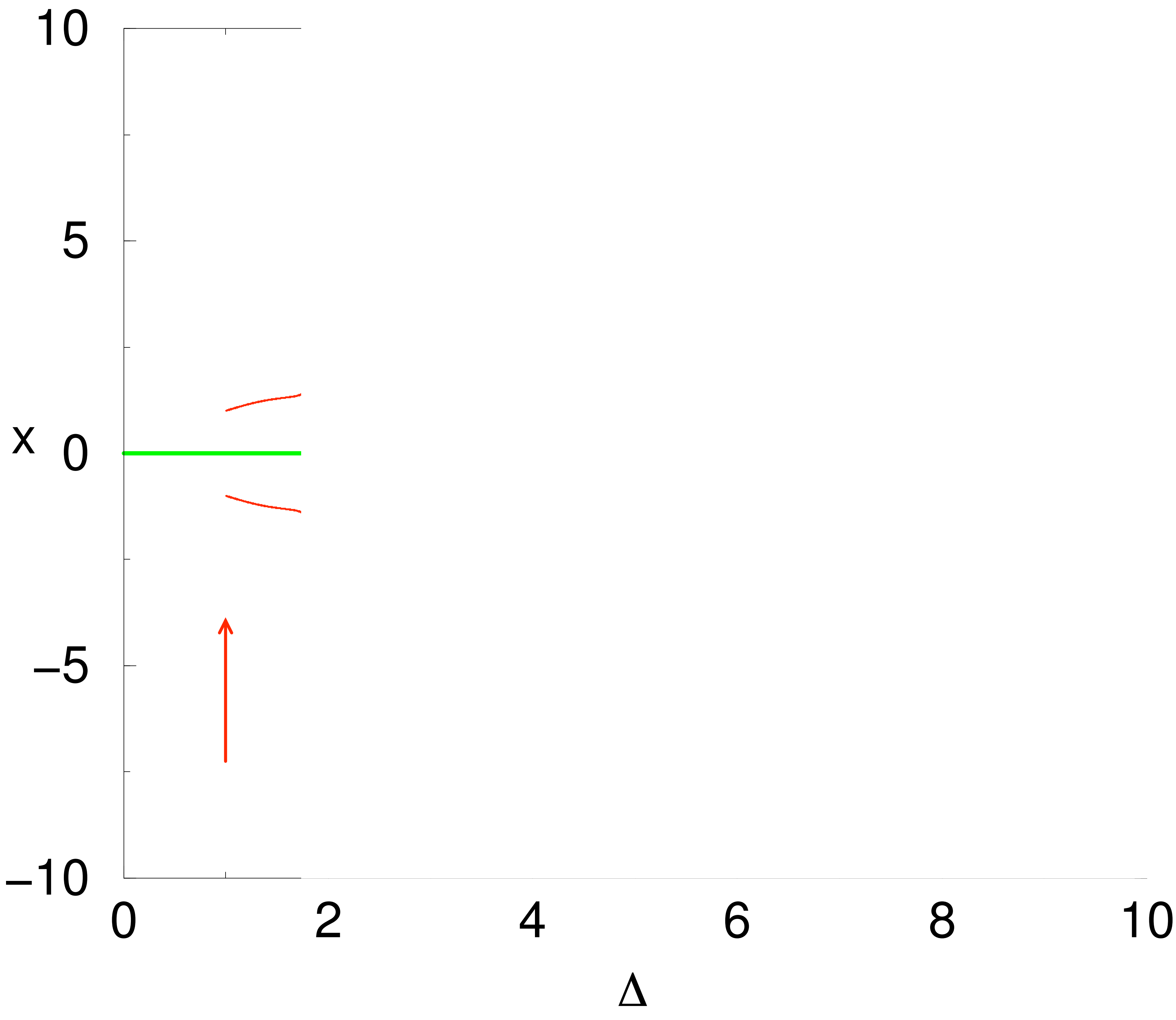

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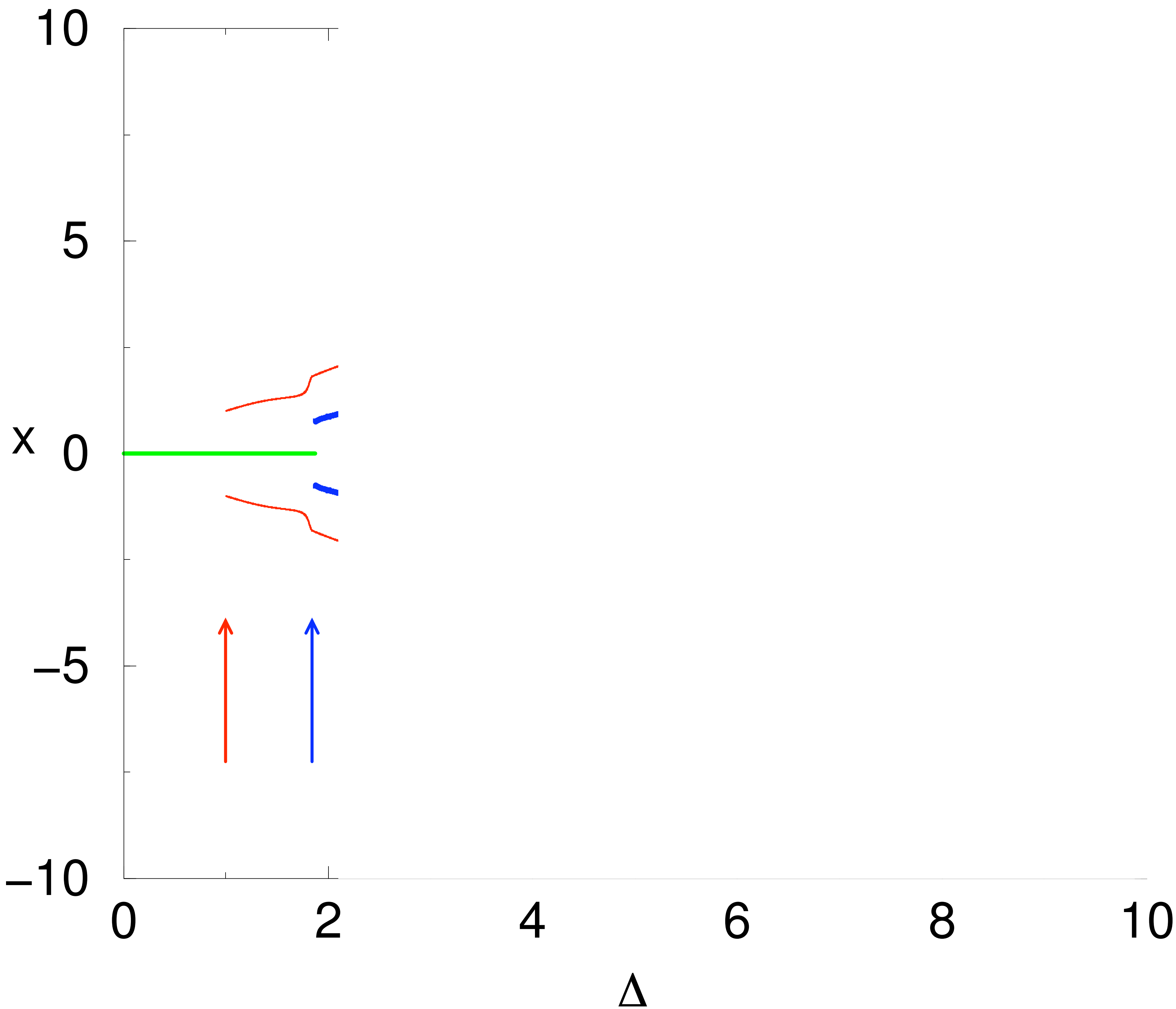

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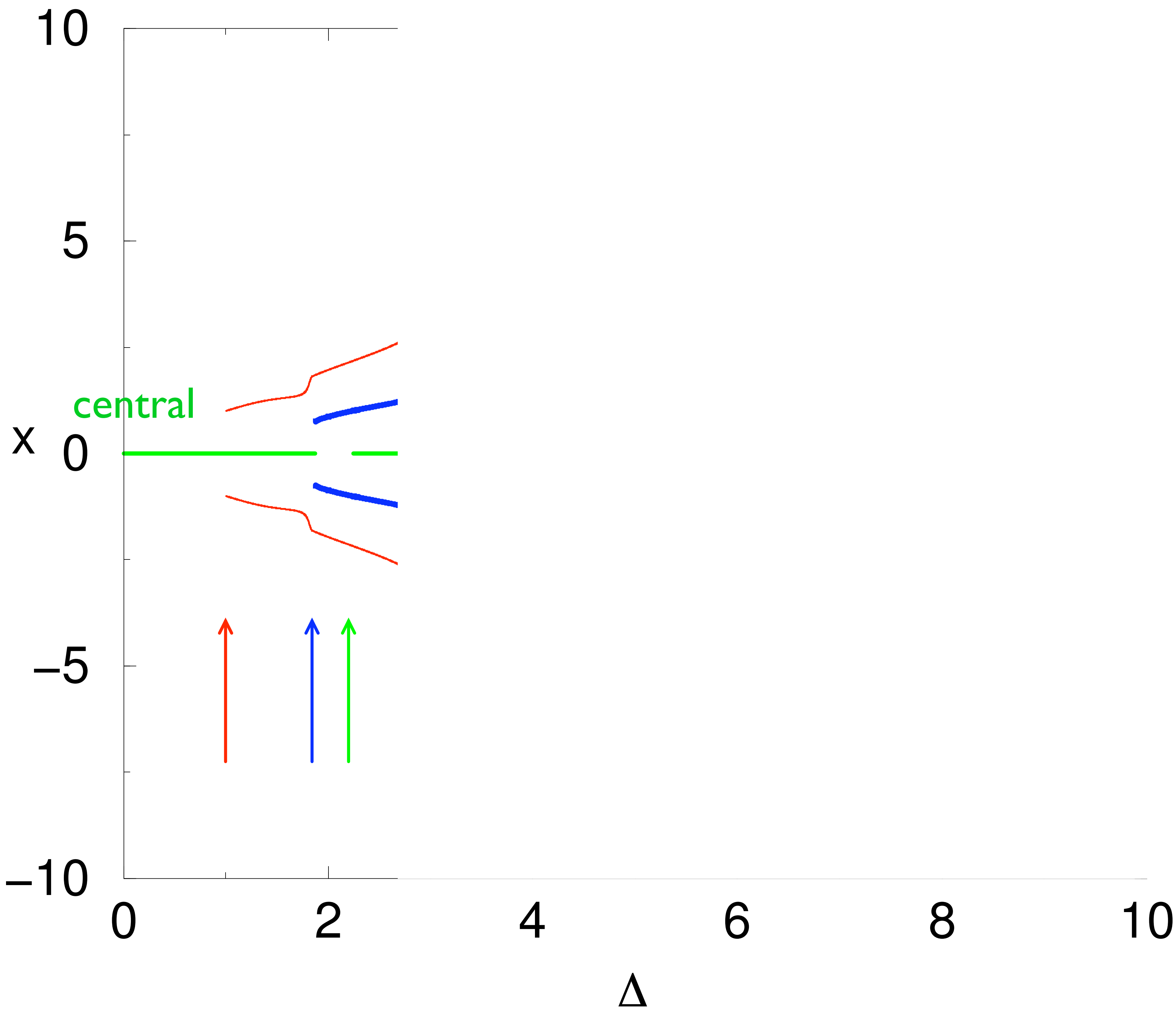
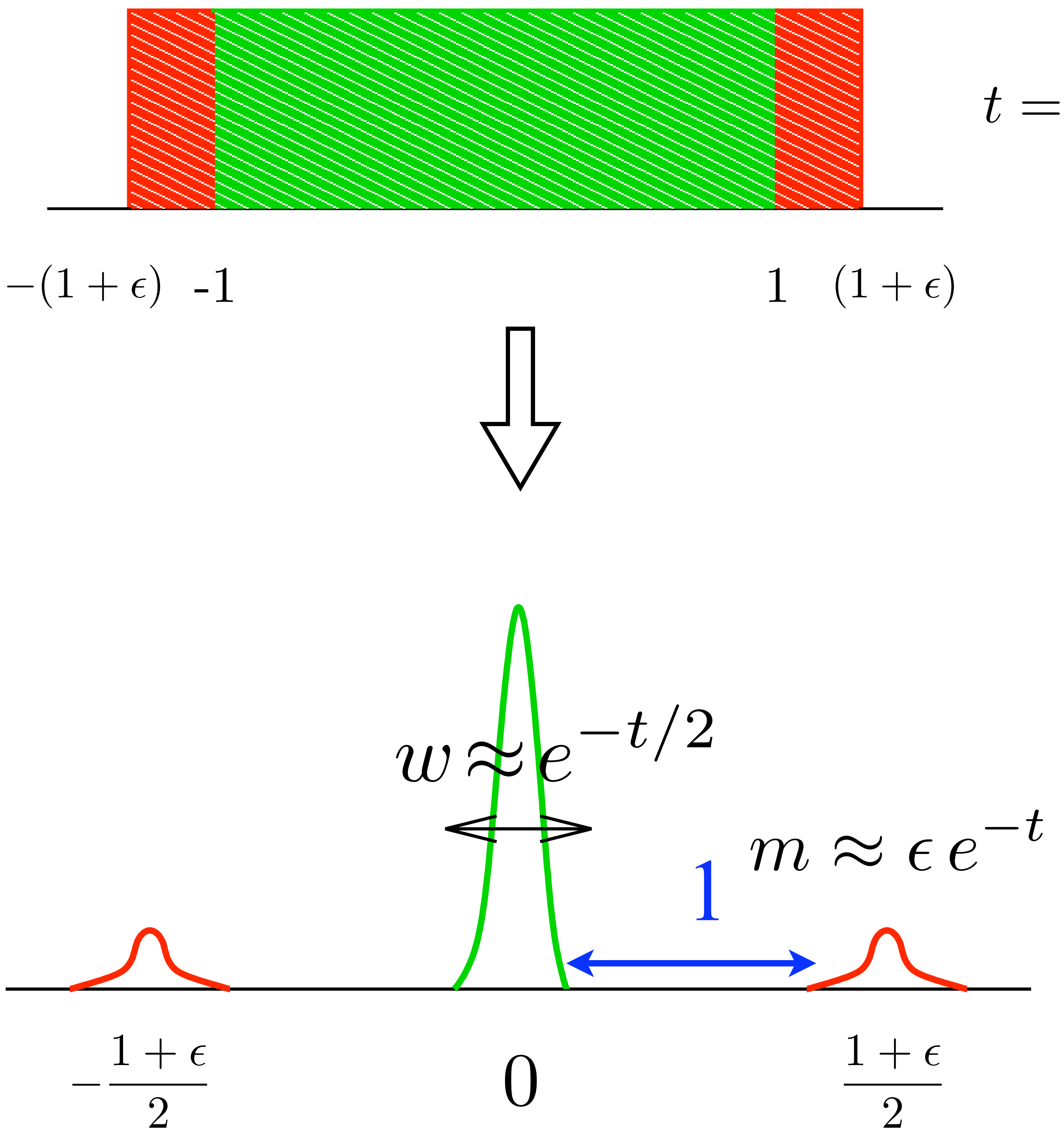


Birth of Extremists



separation:

$w = \epsilon = e^{-t_{\text{sep}}/2}$

$\rightarrow m(t_{\text{sep}}) \propto \epsilon^3$

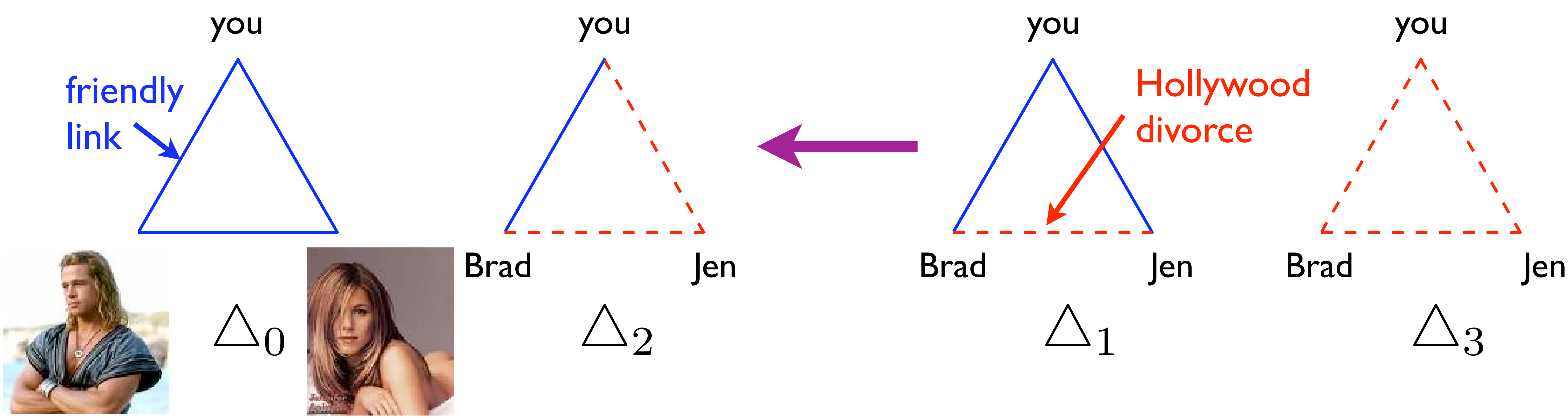
A Possible Realization

1993 Canadian Federal Election

year	PQ	NDP	L	PC	SC	R/CA
1979		26	114	136	6	
1980		32	147	103		
1984		30	40	211		
1988		43	83	169		
1993	54	9	177	2		52
1997	44	21	155	20		60

Dynamics of Social Balance Heider (1944)

Antal et al., PRE **72**, 036121 (05),
Physica D **224**, 130 (06)



unfrustrated/balanced frustrated/imbalanced

Social Balance

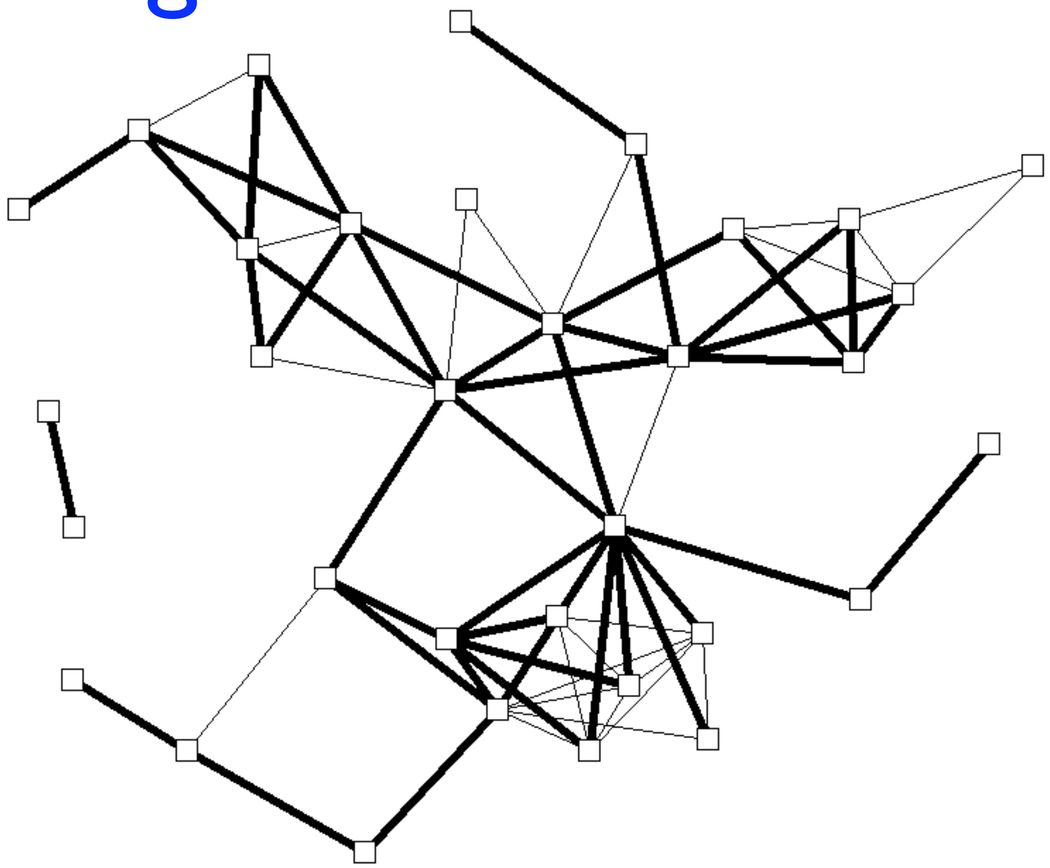
a friend of my friend
an enemy of my enemy } *is my friend;*

a friend of my enemy
an enemy of my friend } *is my enemy.*

Application: Long Beach Street Gangs

Nakamura, Tita, & Krackhardt (2007)

gang relations



cool with

hate

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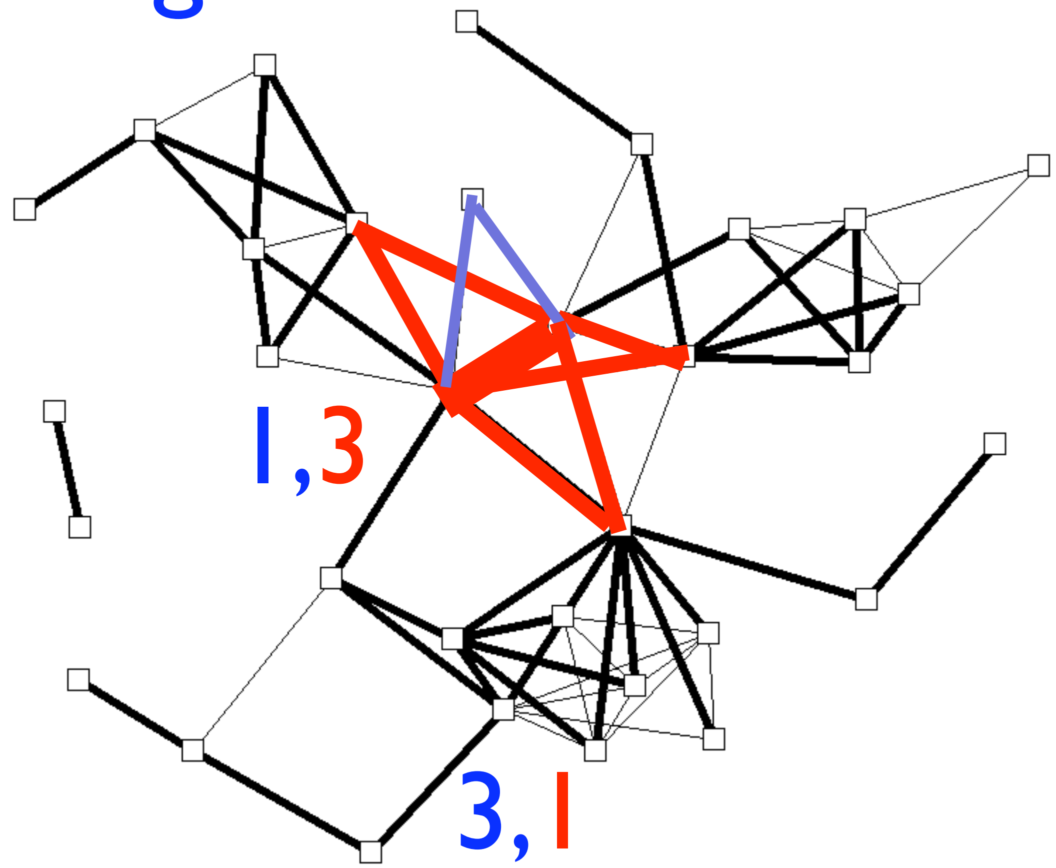
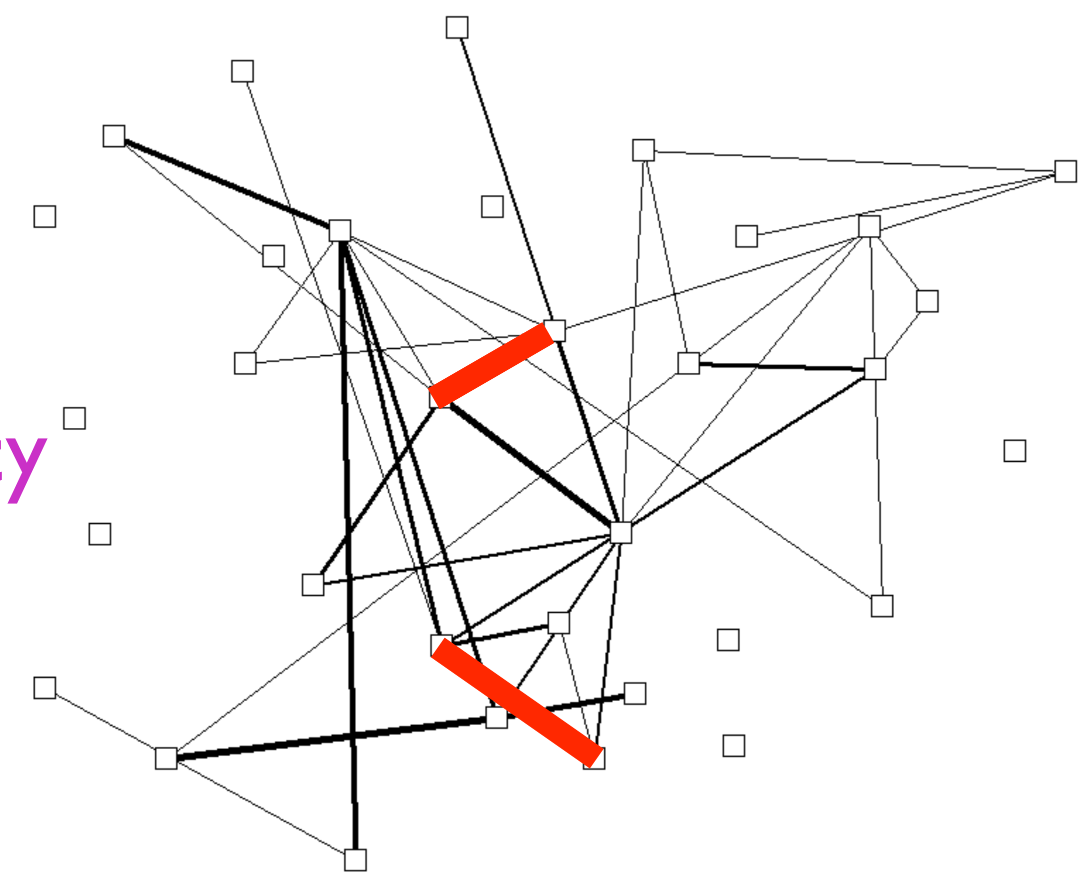


cool with

hate

violence frequency



low incidence

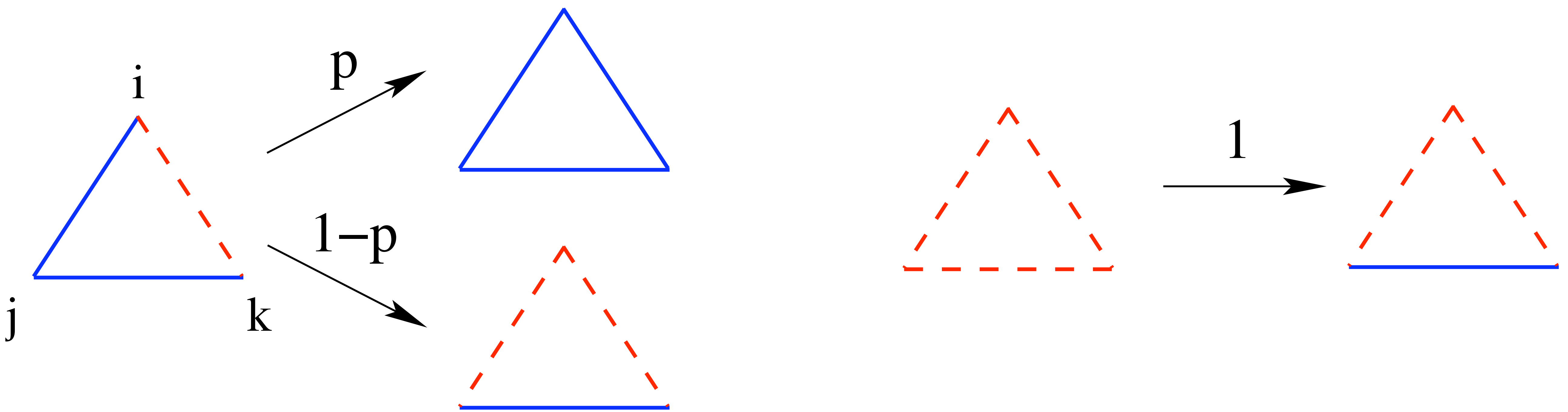
high incidence

Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced

   probability p: unfriendly → friendly;   probability 1-p: friendly → unfriendly



Fundamental parameter p:

- p=1/3: flip a random link in the triad equiprobably
- p>1/3: predisposition toward tranquility
- p<1/3: predisposition toward hostility

The Evolving State

rate equation for the friendly link density:

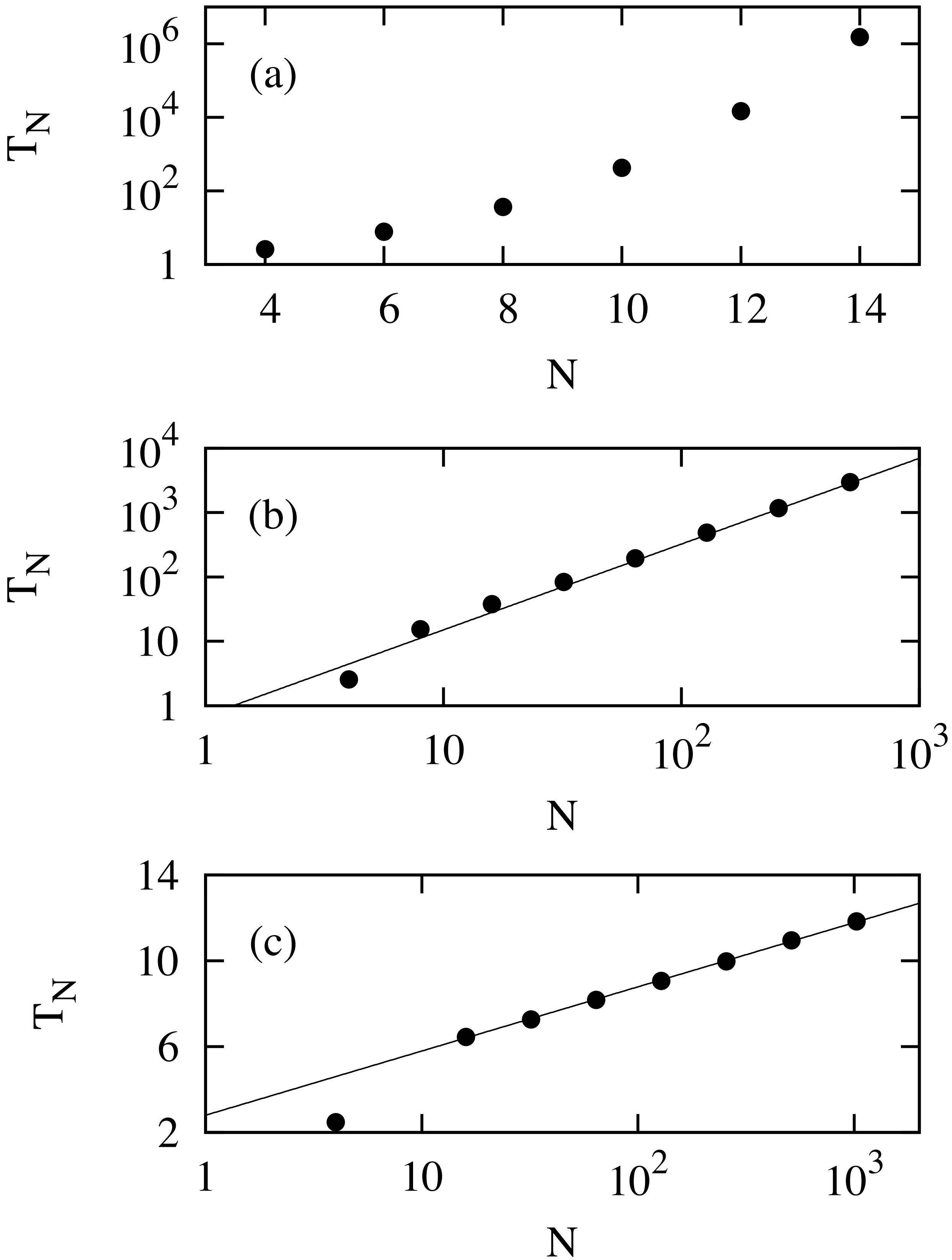
$- \to +$ in $\triangle_1$ $\quad + \to -$ in $\triangle_1$ $\quad - \to +$ in $\triangle_3$

$$\frac{d\rho}{dt} = 3\rho^2(1-\rho)[p-(1-p)] + (1-\rho)^3$$
$$= 3(2p-1)\rho^2(1-\rho) + (1-\rho)^3$$

$$\rho(t) \sim \begin{cases} \rho_\infty + Ae^{-Ct} & p < 1/2; \\ 1 - \dfrac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}} & p = 1/2; \\ 1 - e^{-3(2p-1)t} & p > 1/2. \end{cases}$$

- $p < 1/2$: rapid onset of frustration
- $p = 1/2$: slow relaxation to utopia
- $p > 1/2$: rapid attainment of utopia

Simulations for a Finite Society



$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

$$p = \frac{1}{2}, \qquad T_N \sim N^{4/3}$$

$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p-1}$$

Fate of a Finite Society

p<1/2: effective random walk picture

$D \propto N^2$ $\qquad$ $v \propto N$

$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

p>1/2: inversion of the rate equation

$$u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1}$$

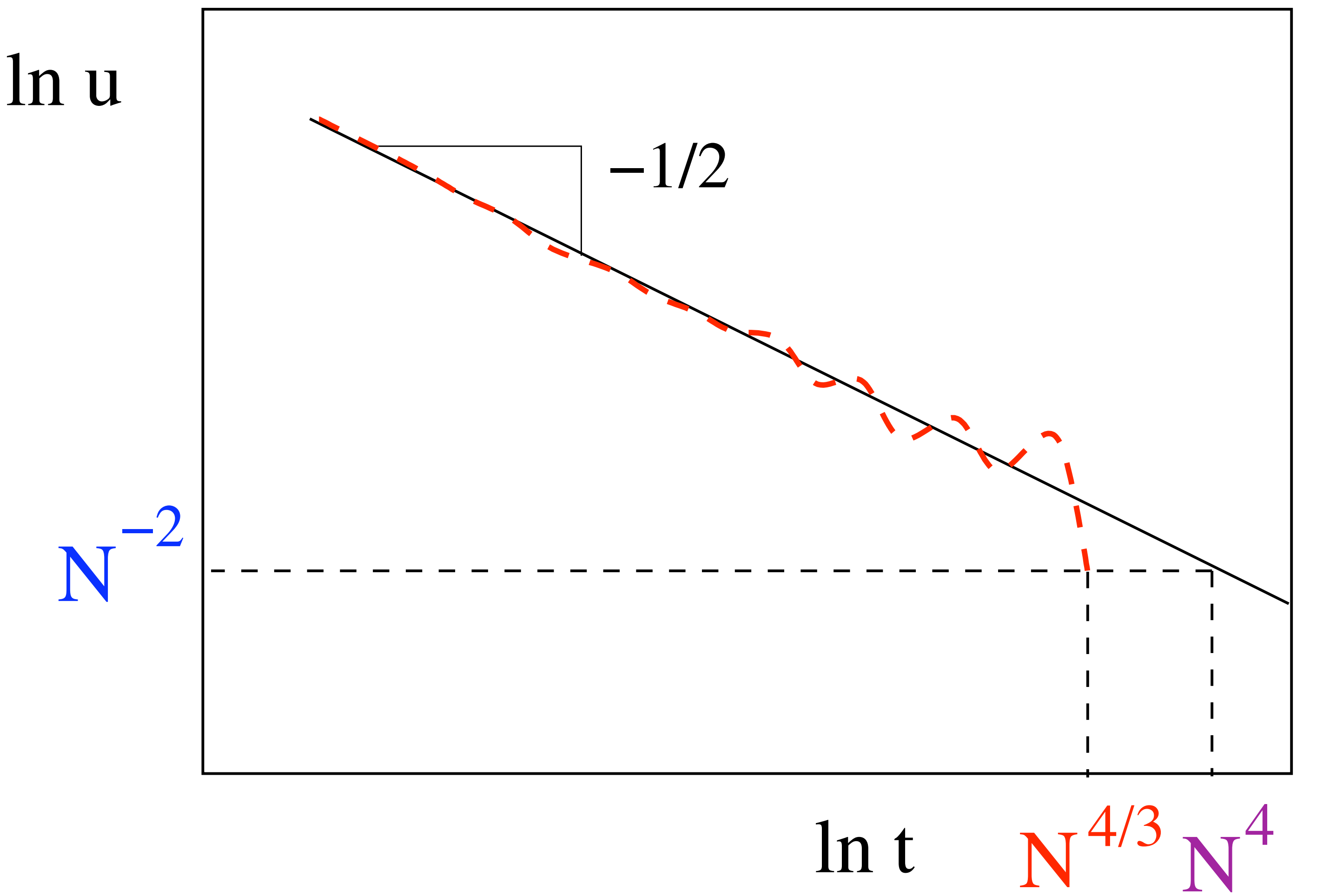
u=1-ρ, the unfriendly link density

p=1/2

naive rate equation estimate:

$$u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \qquad \rightarrow \; T_N \sim N^4$$

incorporating fluctuations as balance is approached:

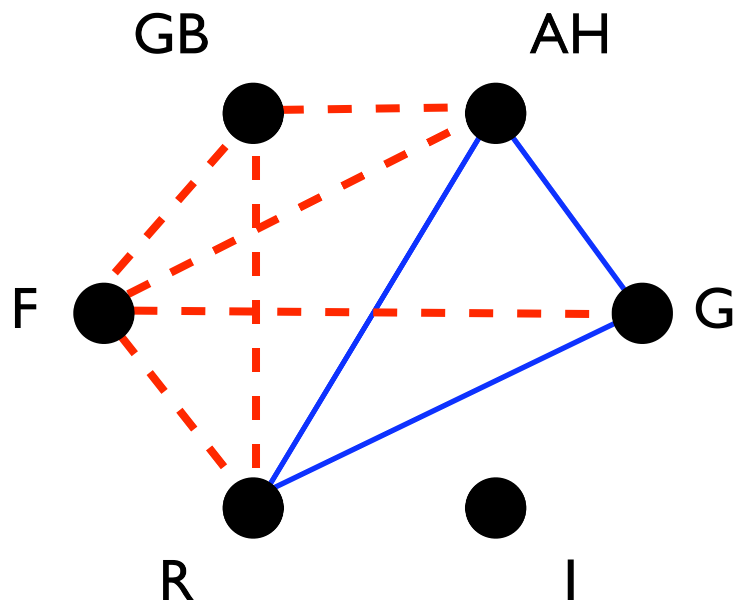


$$U = Lu + \sqrt{L}\,\eta$$
$$\sim \frac{L}{\sqrt{t}} + \sqrt{L}\,t^{1/4}$$

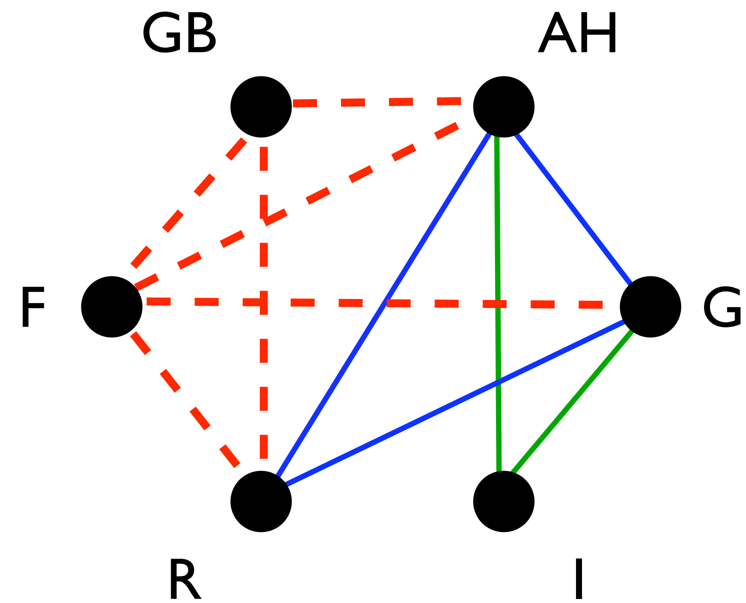
equating the 2 terms in U:

$$T_N \sim L^{2/3} \sim N^{4/3}$$

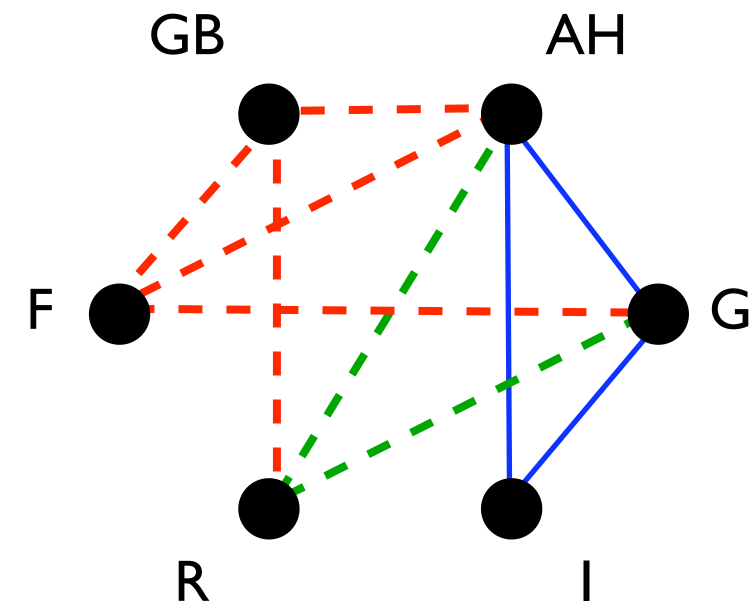
A Historical Lesson



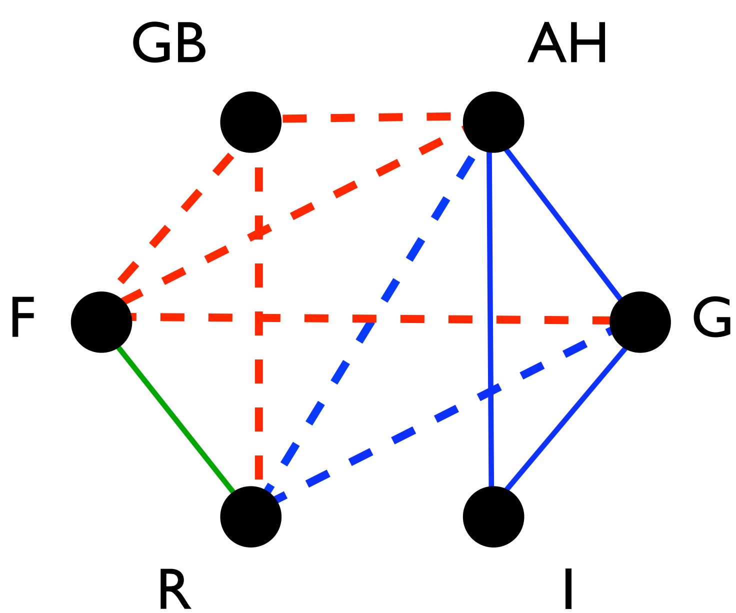
3 Emperor's League 1872-81



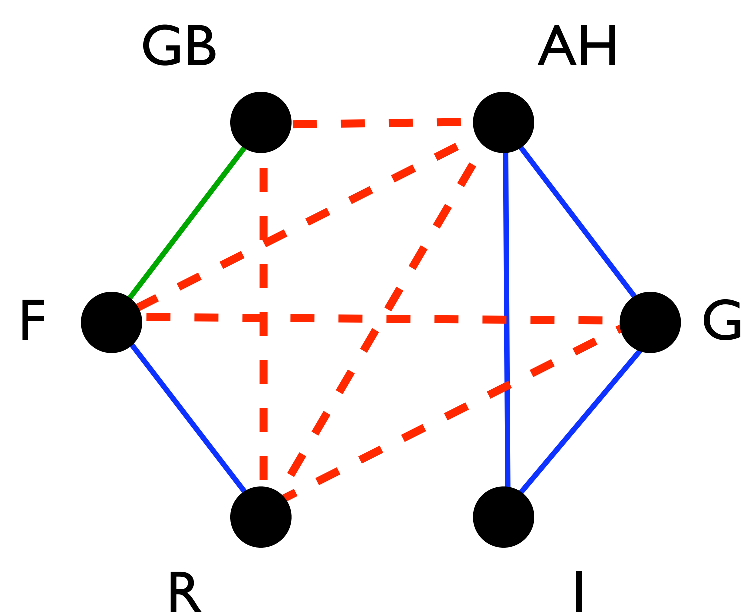
Triple Alliance 1882



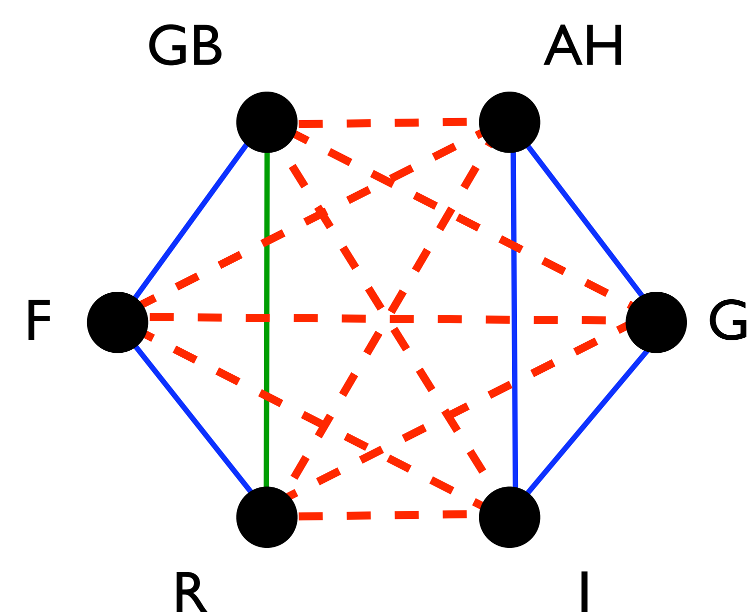
German-Russian Lapse 1890



French-Russian Alliance 1891-94



Entente Cordiale 1904



British-Russian Alliance 1907

Summary & Outlook

Voter model:

paradigmatic, soluble, many untapped applications

Bounded compromise:

a natural mechanism for political fragmentation

self-similar fragmentation as diversity increases

Social Balance:

if we can't all love each other → *social balance*

questions:

incomplete graphs, indifference, continuous interactions,

asymmetric relations, allow ∴ → Machiavellian society,....

Open:

develop empirical connections & predictions